

# JOINT TORSION OF SEVERAL COMMUTING OPERATORS

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## Abstract

We introduce the notion of joint torsion for several commuting operators satisfying a Fredholm condition. This new secondary invariant takes values in the group of invertibles of a field. It is constructed by comparing determinants associated with different filtrations of a Koszul complex. Our notion of joint torsion generalizes the Carey-Pincus joint torsion of a pair of commuting Fredholm operators. As an example, under more restrictive invertibility assumptions, we show that the joint torsion recovers the multiplicative Lefschetz numbers. Furthermore, in the case of Toeplitz operators over the polydisc we provide a link between the joint torsion and the Cauchy integral formula. We will also consider the algebraic properties of the joint torsion. They include a cocycle property, a symmetry property, a triviality property and a multiplicativity property. The proof of these results relies on a quite general comparison theorem for vertical and horizontal torsion isomorphisms associated with certain diagrams of chain complexes.

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## 1. INTRODUCTION

Let us start by presenting a short overview of some results of Richard Carey and Joel Pincus related to their joint torsion invariant. This should serve as sufficient motivation for the introduction of the multivariable generalization which we will discuss afterwards.

The point of depart is a pair of commuting Fredholm operators  $(A, B)$  on some vector space  $E$  over a field  $\mathbb{F}$ . Thus, we assume that all kernels and cokernels are finite dimensional and that the commutator  $[A, B] = 0$  is trivial. The kernel and cokernel of  $A$  then fit in a long exact sequence of vector spaces

$$(1.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(A) \cap \text{Ker}(B) & \longrightarrow & \text{Ker}(A) & \xrightarrow{-B} & \text{Ker}(A) \longrightarrow H_1(A, B) \\ & & & & & & \downarrow \\ \mathcal{E}^A : & & & & & & \\ & & 0 \longleftarrow E/(\text{Im}(A) + \text{Im}(B)) & \longleftarrow & \text{Coker}(A) & \xleftarrow{B} & \text{Coker}(A) \end{array}$$

Here  $H_1(A, B)$  is the first Koszul homology group of the commuting pair  $(A, B)$ . Likewise, the kernel and cokernel of  $B$  fit in another long exact sequence of vector spaces

$$(1.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(A) \cap \text{Ker}(B) & \longrightarrow & \text{Ker}(B) & \xrightarrow{A} & \text{Ker}(B) \longrightarrow H_1(A, B) \\ & & & & & & \downarrow \\ \mathcal{E}^B : & & & & & & \\ & & 0 \longleftarrow E/(\text{Im}(A) + \text{Im}(B)) & \longleftarrow & \text{Coker}(B) & \xleftarrow{-A} & \text{Coker}(B) \end{array}$$

We can then apply a determinant functor to these two long exact sequences of finite dimensional vector spaces. After some canonical identifications, we obtain two possibly distinct isomorphisms

$$\begin{aligned} T(A) \text{ and } T(B) : \det(\text{Ker}(A) \cap \text{Ker}(B)) \otimes \det(E/(\text{Im}(A) + \text{Im}(B))) \\ \longrightarrow \det(H_1(A, B)) \end{aligned}$$

at the level of determinants. The quotient of these two isomorphisms is an automorphism of a one-dimensional vector space and can thus be identified with an invertible number. This number is, up to a sign, the *Carey-Pincus joint torsion* of the commuting pair of Fredholm operators,

$$\tau(A, B) = (-1)^{\nu(A, B)} T(A)^{-1} \circ T(B) \in \mathbb{F}^*$$

Here the exponent  $\nu(A, B) \in \mathbb{N} \cup \{0\}$  is given by dimensions of Koszul homology groups. We refer to the paper [4] for more details on this construction.

Let us pass to a description of some important results. First of all, we outline the relation to the second algebraic  $K$ -group. This relation can be explained by means of the determinant invariant of Larry Brown. The determinant invariant is a homomorphism

$$d : K_2(\mathcal{L}/\mathcal{F}) \rightarrow \mathbb{F}^*$$

from the second algebraic  $K$ -group of the quotient ring  $\mathcal{L}/\mathcal{F}$  to the group of invertibles of the field. Here  $\mathcal{L} := \mathcal{L}(E)$  is the linear operators on the vector space  $E$  and  $\mathcal{F} := \mathcal{F}(E)$  is the ideal of finite rank operators. For more details on the determinant invariant we refer to the book of Jonathan Rosenberg [15] and the papers [3, 10, 16]. The two commuting Fredholm operators  $A$  and  $B$  determine two invertible and commuting elements in the quotient ring  $\mathcal{L}/\mathcal{F}$ . In particular, we get a Steinberg symbol

$$\{q(A), q(B)\} \in K_2(\mathcal{L}/\mathcal{F})$$

in the second algebraic  $K$ -group of the quotient ring. Here  $q : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{F}$  denotes the quotient map. The relation between joint torsion and algebraic  $K$ -theory can now be stated.

**Theorem 1.0.1.** [4, Theorem 2] *The joint torsion of the commuting pair of Fredholm operators  $(A, B)$  agrees with the determinant invariant of the Steinberg symbol  $\{q(A), q(B)\} \in K_2(\mathcal{L}/\mathcal{F})$ . Thus, in formulas we have the identity*

$$\tau(A, B) = d\{q(A), q(B)\}$$

in the group of invertibles  $\mathbb{F}^*$ .

As an important consequence of the above description we get a multiplicativity property for the joint torsion. Indeed, since the Steinberg symbol is bilinear we get that

$$(1.3) \quad \tau(A \cdot C, B) = \tau(A, B) \cdot \tau(C, B)$$

Here  $C$  is an extra Fredholm operator which commutes with  $B$ . It also follows from Theorem 1.0.1 that the joint torsion is invariant under finite rank perturbations.

The next result, which we would like to describe, is a complete calculation of the joint torsion in the case of Toeplitz operators over the disc. This can be thought of as a multiplicative index theorem. We look at two continuous and invertible functions on the circle

$$f, g : S^1 \rightarrow \mathbb{C}^*$$

We will then assume that these two functions extend to holomorphic functions on the disc. The associated Toeplitz operators on Hardy space

$$T_f \text{ and } T_g : H^2(\mathbb{D}^\circ) \rightarrow H^2(\mathbb{D}^\circ)$$

then form a pair of commuting Fredholm operators,  $(T_f, T_g)$ . In particular, we can assign a joint torsion

$$\tau(T_f, T_g) \in \mathbb{C}^*$$

to the pair of Toeplitz operators.

On the other hand, we could look at the behaviour of the meromorphic function  $f/g$  near the zeros and poles. To be precise, the holomorphic functions  $f$  and  $g$  have a finite number of zeros  $\lambda_1, \dots, \lambda_n \in \mathbb{D}^\circ$  in the interior of the disc. The multiplicities of a zero  $\lambda_i \in \mathbb{D}^\circ$  will be denoted by

$$\mu_f(\lambda_i) \text{ and } \mu_g(\lambda_i) \in \mathbb{N} \cup \{0\}$$

for the functions  $f$  and  $g$  respectively. The following limit of quotients

$$c_{\lambda_i}(f, g) = (-1)^{\mu_f(\lambda_i) \cdot \mu_g(\lambda_i)} \lim_{z \rightarrow \lambda_i} \frac{f(z)^{\mu_g(\lambda_i)}}{g(z)^{\mu_f(\lambda_i)}} \in \mathbb{C}^*$$

is then a well-defined invertible number. The product of these limits

$$c(f, g) = \prod_{i=1}^n c_{\lambda_i}(f, g) \in \mathbb{C}^*$$

is known as the *tame symbol* of  $f$  and  $g$ . For more details we refer to the paper [8] by Pierre Deligne.

**Theorem 1.0.2.** [4, Proposition 1] *The joint torsion of the pair of Toeplitz operators  $(T_f, T_g)$  coincides with the tame symbol of the pair of functions  $(f, g)$ . Thus, in formulas we have the identity*

$$\tau(T_f, T_g) = c(f, g)$$

Notice that there are other descriptions available for the tame symbol. For example, this quantity can be expressed as the monodromy of a flat line bundle. See [1, 4, 8].

The very simple question which we will investigate in the present paper can now be formulated:

*"What happens when we replace a pair of commuting Fredholm operators by a commuting tuple?"*

Thus, let us consider a commuting tuple  $A = (A_1, \dots, A_n)$  of linear operators on the vector space  $E$ . This commuting tuple gives rise to a Koszul complex  $K(A)$ . We will think of the Koszul complex as a  $\mathbb{Z}_2$ -graded chain complex and denote the  $\mathbb{Z}_2$ -graded homology group by  $H(A) = H_+(A) \oplus H_-(A)$ . Our first task is to find a replacement for the long exact sequences (1.1) and (1.2). One possibility is to remove one of the operators from the commuting tuple  $A$ , say the operator  $A_i$ . Thus, we could look at the commuting tuple

$$i(A) = (A_1, \dots, \widehat{A_i}, \dots, A_n)$$

for some  $i \in \{1, \dots, n\}$ . It can then be proved that we have a short exact sequence of Koszul complexes

$$(1.4) \quad 0 \longrightarrow K(i(A)) \longrightarrow K(A) \longrightarrow K(i(A))[1] \longrightarrow 0$$

Here the notation "[1]" refers to the operation of changing both the sign of the differential and the grading of a  $\mathbb{Z}_2$ -graded chain complex. In particular, we get a six term exact sequence of even and odd Koszul homology groups,

$$(1.5) \quad \begin{array}{ccccc} H_+(i(A)) & \longrightarrow & H_+(A) & \longrightarrow & H_-(i(A)) \\ & \uparrow A_i & & & \downarrow A_i \\ H_+(i(A)) & \longleftarrow & H_-(A) & \longleftarrow & H_-(i(A)) \end{array}$$

Here the boundary maps are induced by the action of the linear operator  $A_i$  on the Koszul complex of the commuting tuple  $i(A)$ . The six term exact sequences obtained in this fashion can be thought of as analogs of the long exact sequences (1.1) and (1.2).

Let us suppose that the commuting tuple  $i(A)$  is Fredholm. This means that the Koszul homology group  $H(i(A))$  is a finite dimensional vector space. We can then apply a determinant

functor to our six term exact sequence in homology (1.5). After some canonical identifications this gives rise to an isomorphism

$$T_i(A) : \det(H_+(A)) \rightarrow \det(H_-(A))$$

between the determinants of the even and odd Koszul homology groups of the commuting tuple  $A$ .

We could carry out the same construction for some fixed  $j \in \{1, \dots, n\}$ . Thus, if we assume that the commuting tuple  $j(A)$  is Fredholm as well, we get another isomorphism

$$T_j(A) : \det(H_+(A)) \rightarrow \det(H_-(A))$$

between the same one dimensional vector spaces. The quotient of these two isomorphism can therefore be identified with an invertible number

$$\tau_{i,j}(A) = (-1)^{\mu_i(A) + \mu_j(A)} T_j(A)^{-1} \circ T_i(A) \in \mathbb{F}^*$$

This is the *joint torsion transition number* in position  $(i, j)$ . The exponents  $\mu_i(A)$  and  $\mu_j(A) \in \mathbb{N} \cup \{0\}$  are given by appropriate dimensions of Koszul homology groups. It can be proved that our joint torsion recovers the Carey-Pincus joint torsion when the commuting tuple  $A$  consists of a pair of Fredholm operators. The joint torsion transition numbers are the principal subject of the present paper. Let us state our main results. The first one justifies the use of the word "transition" in our definition. Indeed, the joint torsion transition numbers satisfy the same relations as the transition functions of a line bundle.

**Theorem 1.0.3.** *Suppose that  $k \in \{1, \dots, n\}$  is an extra number such that the commuting tuple  $k(A) = (A_1, \dots, \widehat{A_k}, \dots, A_n)$  is Fredholm. The joint torsion transition numbers then satisfy the cocycle property*

$$\tau_{i,j}(A) = \tau_{j,i}(A)^{-1} \quad \tau_{i,j}(A) \cdot \tau_{j,k}(A) = \tau_{i,k}(A)$$

The second one is a triviality property. It says that the joint torsion transition number in position  $(i, j)$  is equal to the identity when the commuting tuples  $i(A)$  and  $j(A)$  are Fredholm for a trivial reason. For example, in the case of a commuting pair of operators it means that the joint torsion is trivial when the vector space is finite dimensional. As another example, the theorem implies that the joint torsion transition number in position  $(i, j)$  is equal to one, when  $A_k \in \mathcal{L}(E)$  is a Fredholm operator for some  $k \neq i, j$ .

**Theorem 1.0.4.** *The joint torsion transition number in position  $(i, j)$  is trivial when the Koszul homology group  $H((ij)(A))$  is finite dimensional. Here we let*

$$(ij)(A) := (A_1, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_n)$$

*denote the commuting tuple obtained from  $A$  by removing both of the operators  $A_i$  and  $A_j$ .*

The final result is a multiplicativity property. It can be understood as an analog of the multiplicativity property for the Carey-Pincus joint torsion stated in (1.3). We let  $B = (A_1, \dots, B_m, \dots, A_n)$  be another commuting tuple which only differs from  $A = (A_1, \dots, A_n)$  in the  $m^{\text{th}}$  coordinate. By the product of  $A$  and  $B$  we will then understand the commuting tuple

$$A \cdot B = (A_1, \dots, A_m \cdot B_m, \dots, A_n)$$

Notice that we do not assume that the operators  $A_m$  and  $B_m$  commute.

**Theorem 1.0.5.** *Suppose that two of the three joint torsion transition numbers  $\tau_{i,j}(A)$ ,  $\tau_{i,j}(B)$  and  $\tau_{i,j}(A \cdot B)$  make sense. Then the third one is also well-defined and related to the two others by the multiplicativity relation*

$$\tau_{i,j}(A) \cdot \tau_{i,j}(B) = \tau_{i,j}(A \cdot B)$$

It might be worthwhile to discuss some aspects of the proofs of Theorem 1.0.4 and Theorem 1.0.5. The main tool is a comparison result for determinants of certain triangles of chain complexes. These triangles appear as the rows and columns of a larger diagram of chain complexes. This comparison theorem lies at the technical core of the paper and we spend some time giving a detailed proof. It should be noted that there most certainly exists a link between our results and the general construction of determinant functors of triangulated categories. See [2, 13]. Making this link explicit would furnish the abstract theory of determinants with a concrete operator theoretic application. Furthermore, it would help conceptualizing the direct approach which we apply in this article.

There are at least two other important issues which we do not treat in this paper, but which we hope to address in the future.

The first one is the relation between the joint torsion transition numbers and algebraic  $K$ -theory. For example, suppose that the Koszul homology group  $H(A) = \{0\}$  is trivial. In this case, we give a formula for the joint torsion transition numbers in terms of quotients of determinants. See Theorem 3.4.1. The appearance of this multiplicative Lefschetz number suggests that the joint torsion transition numbers could be obtained as values of the determinant invariant on the second algebraic  $K$ -group. This is also supported by Theorem 1.0.1 which states that this is the case in the low-dimensional situation. The  $K$ -theoretic interpretation of the joint torsion is currently under investigation.

The second issue is the extension of the Carey-Pincus multiplicative index theorem to the case of Toeplitz operators over the polydisc. See Theorem 1.0.2. This task is harder than finding a  $K$ -theoretic interpretation of the joint torsion. Indeed, it is not completely clear what the correct replacement of the tame symbol should be. However, let  $f \in \mathcal{A}(U^n)$  be an invertible element of the polydisc algebra. We can then compute the joint torsion transition number in position  $(1, j)$  of the commuting tuple

$$T_\alpha = (T_f, T_{z_1 - \alpha_1}, \dots, T_{z_n - \alpha_n})$$

of Toeplitz operators. Here  $\alpha_1, \dots, \alpha_n \in \mathbb{D}^\circ := U$  are complex numbers in the interior of the disc and  $z_1, \dots, z_n : \mathbb{T}^n \rightarrow \mathbb{C}$  are the coordinate functions on the  $n$ -torus. In this case, the joint torsion is simply given by the evaluation of  $f \in \mathcal{A}(U^n)$  at the point  $\alpha = (\alpha_1, \dots, \alpha_n) \in U^n$ ,

$$\tau_{1,j}(T_\alpha) = f(\alpha) \in \mathbb{C}^*$$

This result is obtained in Theorem 3.5.2.

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## 2. TORSION ISOMORPHISMS

In this section we will construct torsion isomorphisms in different contexts. The first subsection is concerned with the most basic situation. We look at an odd endomorphism of a  $\mathbb{Z}_2$ -graded vector space which satisfies an exactness condition. We then associate a determinant to such an endomorphism. The second subsection is concerned with determinants of odd exact triangles of finite dimensional vector spaces. We use the construction of the first subsection to give a definition of a determinant in this situation. The last subsection is concerned with determinants of odd triangles of chain complexes which satisfy a homotopy exactness condition. Furthermore, these chain complexes are supposed to have finite dimensional homology. We use the construction of the second subsection to give a definition of a determinant in this situation.

It should be noted that the material of this section is rather basic but also absolutely essential for the rest of the paper. We should also stress the relation to the construction of determinant functors in various settings one more time. Thus, we refer to the papers [2, 9, 11, 12, 13]. However, our exposition is very simple minded and does not rely on any of these more elaborated results.

**2.1. The torsion isomorphism of an odd exact endomorphism.** Let  $\mathbb{F}$  be a field and let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{F}$ . We let  $\det(V)$  denote the top part of the exterior algebra over  $V$ ,

$$\det(V) := \Lambda_n(V) \quad n = \dim(V)$$

This one-dimensional vector space will be referred to as the *determinant of  $V$* . Furthermore, we let  $V^* := \operatorname{Hom}(V, \mathbb{F})$  denote the dual of the vector space  $V$ .

Now, let  $E_+$  and  $E_-$  be two vector spaces over  $\mathbb{F}$  of the same finite dimension. Let  $\sigma : E_+ \rightarrow E_-$  be an isomorphism. We will then let

$$\det(\sigma) : \det(E_+) \rightarrow \det(E_-)$$

denote the isomorphism obtained by functoriality of the exterior power. This isomorphism will be referred to as the *determinant of  $\sigma$* . In the case where  $E_+ = E_-$  we can identify the determinant of  $\sigma$  with an invertible element in the field  $\mathbb{F}$ .

Let us look at the  $\mathbb{Z}_2$ -graded finite dimensional vector space

$$E := E_+ \oplus E_-$$

given by the direct sum of  $E_+$  and  $E_-$ .



Let  $\alpha : E \rightarrow E$  be an odd endomorphism, thus  $\alpha \in \text{End}_-(E)$  is given by a matrix

$$\alpha = \begin{pmatrix} 0 & \alpha_- \\ \alpha_+ & 0 \end{pmatrix} : E_+ \oplus E_- \rightarrow E_+ \oplus E_-$$

relative to the decomposition  $E = E_+ \oplus E_-$ .

**Definition 2.1.1.** *We will say that  $\alpha \in \text{End}_-(E)$  is exact when the kernel of  $\alpha_+$  agrees with the image of  $\alpha_-$  and vice versa, thus*

$$\text{Ker}(\alpha_+) = \text{Im}(\alpha_-) \quad \text{Ker}(\alpha_-) = \text{Im}(\alpha_+)$$

Suppose that  $\alpha \in \text{End}_-(E)$  is exact and let us choose a pseudo-inverse  $\alpha_-^\dagger : E_+ \rightarrow E_-$  of  $\alpha_- : E_- \rightarrow E_+$ . Recall that this corresponds to the choice of algebraic decompositions

$$E_+ \cong \text{Im}(\alpha_-) \oplus Q_- \quad \text{and} \quad E_- \cong \text{Ker}(\alpha_-) \oplus C_-$$

It then follows from the exactness of  $\alpha \in \text{End}_-(E)$  that the linear map

$$\alpha_+ + \alpha_-^\dagger : E_+ \rightarrow E_-$$

is an isomorphism.

**Definition 2.1.2.** *By the torsion isomorphism of  $\alpha \in \text{End}_-(E)$  we will understand the determinant of the isomorphism  $\alpha_+ + \alpha_-^\dagger : E_+ \rightarrow E_-$ . The torsion isomorphism of  $\alpha$  will be denoted by  $T(\alpha)$ , thus*

$$T(\alpha) := \det(\alpha_+ + \alpha_-^\dagger) \in \text{Hom}(\det(E_+), \det(E_-)) - \{0\}$$

Notice that the torsion isomorphism is not a non-zero element in the field  $\mathbb{F}$  but only a non-zero vector in a one-dimensional vector space over  $\mathbb{F}$ .

**Lemma 2.1.3.** *The torsion isomorphism of  $\alpha$  is independent of the choice of pseudo-inverse of  $\alpha_-$ .*

*Proof.* Let  $\alpha_-^* : E_+ \rightarrow E_-$  be a different choice of a pseudo inverse for  $\alpha_- : E_- \rightarrow E_+$ . Let us use the notation

$$\begin{aligned} e &= \alpha_-^\dagger \alpha_- \in \mathcal{L}(E_-) & p &= \alpha_- \alpha_-^\dagger \in \mathcal{L}(E_+) \\ f &= \alpha_-^* \alpha_- \in \mathcal{L}(E_-) & q &= \alpha_- \alpha_-^* \in \mathcal{L}(E_+) \end{aligned}$$

for the idempotents associated with the different choices of pseudo-inverses.

We then have the relation

$$\alpha_+ + \alpha_-^* = \alpha_+ + f \alpha_-^\dagger q = (1 - e + f)(\alpha_+ + \alpha_-^\dagger)(1 - p + q)$$

between the different isomorphisms. In particular, we get the relation

$$\det(\alpha_+ + \alpha_-^*) = \det(1 - e + f) \cdot \det(\alpha_+ + \alpha_-^\dagger) \cdot \det(1 - p + q)$$

between the different determinants. The result of the lemma is now a consequence of the identities

$$\det(1 - e + f) = 1 = \det(1 - p + q)$$

which can be easily verified.  $\square$

**2.2. The torsion isomorphism of an odd exact triangle.** Let us consider an odd triangle of vector spaces

$$V : V^1 \xrightarrow{v^1} V^2 \xrightarrow{v^2} V^3 \xrightarrow{v^3} V^1$$

This means that the vector spaces

$$V^i = V_+^i \oplus V_-^i$$

are  $\mathbb{Z}_2$ -graded for all  $i \in \{1, 2, 3\}$  and that the linear maps

$$v^1 : V^1 \rightarrow V^2 \quad v^2 : V^2 \rightarrow V^3 \quad \text{and} \quad v^3 : V^3 \rightarrow V^1$$

are odd.

**Definition 2.2.1.** *We will say that the odd triangle  $V$  is exact, when the sequence of six terms*

$$\begin{array}{ccccc} V_+^1 & \xrightarrow{v_+^1} & V_-^2 & \xrightarrow{v_-^2} & V_+^3 \\ v_-^3 \uparrow & & & & v_+^3 \downarrow \\ V_-^3 & \xleftarrow{v_+^2} & V_+^2 & \xleftarrow{v_-^1} & V_-^1 \end{array}$$

*is exact.*

We can form a  $\mathbb{Z}_2$ -graded vector space out of the odd triangle  $V$ . We will also use the letter  $V := V_+ \oplus V_-$  for this vector space. It is given by the positive and negative components

$$(2.1) \quad V_+ := V_+^1 \oplus V_+^2 \oplus V_+^3 \quad \text{and} \quad V_- := V_-^1 \oplus V_-^2 \oplus V_-^3$$

We then get an odd endomorphism

$$(2.2) \quad v : V \rightarrow V \quad v := \begin{pmatrix} 0 & v_- \\ v_+ & 0 \end{pmatrix}$$

defined by the matrices

$$v_+ := \begin{pmatrix} 0 & 0 & v_+^3 \\ v_+^1 & 0 & 0 \\ 0 & v_+^2 & 0 \end{pmatrix} \quad \text{and} \quad v_- := \begin{pmatrix} 0 & 0 & v_-^3 \\ v_-^1 & 0 & 0 \\ 0 & v_-^2 & 0 \end{pmatrix}$$

Here we use the decompositions of  $V_+$  and  $V_-$  given by (2.1).

The following lemma is then an immediate consequence of the definitions.

**Lemma 2.2.2.** *The odd homomorphism  $v \in \text{End}_-(V)$  is exact if and only if the odd triangle  $V$  is exact.*

Suppose that our odd triangle  $V$  is exact and that the vector space  $V^i$  is finite dimensional for all  $i \in \{1, 2, 3\}$ . By combining Lemma 2.2.2 and Lemma 2.1.3 we can give a definition of the torsion isomorphism associated to  $V$ .

**Definition 2.2.3.** *By the torsion isomorphism of the odd exact triangle  $V$  we will understand the torsion isomorphism of the odd exact endomorphism  $v \in \text{End}_-(V)$ . The torsion isomorphism of  $V$  will be denoted by  $T(V)$ , thus by definition*

$$T(V) := T(v) \in \text{Hom}(\det(V_+), \det(V_-)) - \{0\}$$

**2.3. The torsion isomorphism of an odd homotopy exact triangle.** Let us consider a  $\mathbb{Z}_2$ -graded chain complex,  $X$ . Thus,  $X = X_+ \oplus X_-$  is a  $\mathbb{Z}_2$ -graded vector space equipped with an odd homomorphism

$$d : X \rightarrow X \quad d := \begin{pmatrix} 0 & d_- \\ d_+ & 0 \end{pmatrix}$$

with square equal to zero

$$d^2 = \begin{pmatrix} d_-d_+ & 0 \\ 0 & d_-d_+ \end{pmatrix} = 0$$

The homology of  $X$  is given by the  $\mathbb{Z}_2$ -graded vector space

$$H(X) = \text{Ker}(d)/\text{Im}(d) = H_+(X) \oplus H_-(X)$$

where the components are defined by

$$H_+(X) := \text{Ker}(d_+)/\text{Im}(d_-) \quad H_-(X) := \text{Ker}(d_-)/\text{Im}(d_+)$$

We will refer to the odd homomorphism  $d : X \rightarrow X$  as the differential on  $X$ .

**Definition 2.3.1.** *By an odd triangle of chain complexes we will understand an odd triangle of vector spaces*

$$X : X^1 \xrightarrow{v^1} X^2 \xrightarrow{v^2} X^3 \xrightarrow{v^3} X^1$$

such that

- (1) *Each  $\mathbb{Z}_2$ -graded vector space  $X^i$  is a  $\mathbb{Z}_2$ -graded chain complex.*
- (2) *The odd linear maps*

$$v^1 : X^1 \rightarrow X^2 \quad v^2 : X^2 \rightarrow X^3 \quad v^3 : X^3 \rightarrow X^1$$

*are odd chain maps. This means that they are odd homomorphisms of  $\mathbb{Z}_2$ -graded vector spaces which anti-commute with the differentials.*

Let us fix an odd triangle of chain complexes

$$X : X^1 \xrightarrow{v^1} X^2 \xrightarrow{v^2} X^3 \xrightarrow{v^3} X^1$$

By passing to homology we get an induced odd triangle of vector spaces

$$H(X) : H(X^1) \xrightarrow{v^1} H(X^2) \xrightarrow{v^2} H(X^3) \xrightarrow{v^3} H(X^1)$$

We will now give a sufficient condition for the exactness of this odd triangle at the level of homology.

**Definition 2.3.2.** *We will say that the odd triangle of chain complexes  $X$  is homotopy exact when there exist odd linear maps*

$$t^1 : X^2 \rightarrow X^1 \quad t^2 : X^3 \rightarrow X^2 \quad t^3 : X^1 \rightarrow X^3$$

*which satisfy the conditions*

- (1) The maps define chain homotopies between 0 and the squares of the odd chain maps associated with  $X$ . Thus, we have the identity

$$d^{i-1}t^{i-1} + t^{i-1}d^i = v^{i+1}v^i : X^i \rightarrow X^{i-1}$$

for all  $i \in \{1, 2, 3\}$ . Here the notation  $d^i : X^i \rightarrow X^i$  refers to the differential on  $X^i$ . Notice that we calculate with the indices modulo three.

- (2) The maps define a homotopy decomposition of the chain complexes in the sense that the chain maps

$$v^{i-1}t^{i-1} + t^i v^i : X^i \rightarrow X^i$$

induce the identity at the level of homology for all  $i \in \{1, 2, 3\}$ . Remark that the chain map property follows from the above chain homotopy condition.

We will refer to the opposite odd triangle of vector spaces

$$X^\dagger : X^1 \xleftarrow{t^1} X^2 \xleftarrow{t^2} X^3 \xleftarrow{t^3} X^1$$

as a homotopy for  $X$ .

As promised above we then have the following lemma.

**Lemma 2.3.3.** *If the odd triangle of chain complexes  $X$  is homotopy exact, then the odd triangle of vector spaces  $H(X)$  is exact.*

*Proof.* We need to prove that the induced sequence of six terms

$$\begin{array}{ccccc} H_+(X^1) & \xrightarrow{v^1} & H_-(X^2) & \xrightarrow{v^2} & H_+(X^3) \\ \uparrow v^3 & & & & \downarrow v^3 \\ H_-(X^3) & \xleftarrow{v^2} & H_+(X^2) & \xleftarrow{v^1} & H_-(X^1) \end{array}$$

is exact.

Let  $i \in \{1, 2, 3\}$ . First of all the identity

$$v^{i+1}v^i = 0 : H(X^i) \rightarrow H(X^{i-1})$$

follows by noting that  $t^{i-1} : X^i \rightarrow X^{i-1}$  defines a homotopy between  $v^{i+1}v^i : X^i \rightarrow X^{i-1}$  and zero.

Now, assume that  $v^i[x] = 0$  for some  $[x] \in H(X^i)$ . Let  $x \in X^i$  be a representative of  $[x] \in H(X^i)$  and let  $y \in X^{i+1}$  be an element with  $d^{i+1}(y) = v^i(x)$ . Notice that  $d^i(x) = 0$  by assumption. We define an element  $z \in X^{i-1}$  by the formula

$$z := v^{i+1}(y) + t^{i-1}(x)$$

We then have that

$$\begin{aligned} d^{i-1}(z) &= (d^{i-1}v^{i+1})(y) + (d^{i-1}t^{i-1})(x) \\ &= -(v^{i+1}d^{i+1})(y) + (v^{i+1}v^i)(x) \\ &= 0 \end{aligned}$$

Thus,  $z \in X^{i-1}$  defines a class  $[z] \in H(X^{i-1})$ . We can then calculate as follows

$$\begin{aligned} v^{i-1}[z] &= [(v^{i-1}v^{i+1})(y) + (v^{i-1}t^{i-1})(x)] \\ &= [(t^i d^{i+1})(y) + (v^{i-1}t^{i-1})(x)] \\ &= [(t^i v^i)(x) + (v^{i-1}t^{i-1})(x)] \\ &= [x] \end{aligned}$$

in the homology group  $H(X^i)$ . This proves the lemma.  $\square$

Suppose that  $X$  is an odd homotopy exact triangle of chain complexes. Furthermore, suppose that the induced odd exact triangle  $H(X)$  consists of finite dimensional vector spaces.

**Definition 2.3.4.** *By the torsion isomorphism of the odd homotopy exact triangle  $X$  we will understand the torsion isomorphism of the odd exact triangle induced by  $X$  at the level of homology. Thus, by definition*

$$T(X) := T(H(X)) \in \text{Hom}(\det(H_+(X)), \det(H_-(X))) - \{0\}$$

Here we recall that  $H(X)$  also denotes the  $\mathbb{Z}_2$ -graded vector space given by the components

$$\begin{aligned} H_+(X) &:= H_+(X^1) \oplus H_+(X^2) \oplus H_+(X^3) \quad \text{and} \\ H_-(X) &:= H_-(X^1) \oplus H_-(X^2) \oplus H_-(X^3) \end{aligned}$$

See Section 2.2.

### 3. JOINT TORSION OF COMMUTING OPERATORS

The main focus of this section lies on the construction of the joint torsion transition numbers of a commuting tuple of linear operators satisfying a Fredholm condition. In the first subsection we recall the definition of the Koszul complex of a commuting tuple of linear operators. In the second section we will review some results from the Fredholm theory in several variables. We will do this in a purely algebraic context. In the third section we construct the joint torsion transition numbers and prove some of their basic properties. This section lies at the heart of the present article. In the fourth and fifth section we compute the joint torsion transition numbers in some concrete examples. The first example is concerned with the situation where the Koszul homology of our commuting tuple is trivial. In this setup we obtain a multiplicative Lefschetz number as the value of our joint torsion invariant. The framework for the second example is the theory of Toeplitz operators over the polydisc. The calculation which we present here provides a link between the joint torsion transition numbers and the Cauchy integral formula for holomorphic functions on the polydisc. In particular, we get an alternative proof of the fact, that a continuous function on the torus has at most one holomorphic extension to the polydisc.

Let us start by indicating a sign convention which we will use throughout. Let  $V, W, V_1, \dots, V_m$  be finite dimensional vector spaces and let  $\sigma \in \Sigma_m$  be a permutation. We can then form the finite dimensional vector spaces

$$F_V := V \oplus (V_1 \oplus \dots \oplus V_m) \quad \text{and} \quad F_W := W \oplus (V_{\sigma(1)} \oplus \dots \oplus V_{\sigma(m)})$$

We will identify the one-dimensional vector spaces

$$\mathrm{Hom}(\det(F_V), \det(F_W)) \quad \text{and} \quad \mathrm{Hom}(\det(V), \det(W))$$

using a string of isomorphisms. To be precise, we have that

$$\begin{aligned} \mathrm{Hom}(\det(F_V), \det(F_W)) &\cong \det(F_V)^* \otimes \det(F_W) \\ &\cong \det(V)^* \otimes \det(V_1)^* \otimes \dots \otimes \det(V_m)^* \\ &\quad \otimes \det(W) \otimes \det(V_{\sigma(1)}) \otimes \dots \otimes \det(V_{\sigma(m)}) \\ &\cong \det(V)^* \otimes \det(W) \\ &\cong \mathrm{Hom}(\det(V), \det(W)) \end{aligned} \tag{3.1}$$

Here the second last isomorphism is given by

$$\gamma \otimes (\gamma_1 \otimes \dots \otimes \gamma_m) \otimes x \otimes (x_1 \otimes \dots \otimes x_m) \mapsto (\gamma \otimes x) \cdot \gamma_{\sigma(1)}(x_1) \cdot \dots \cdot \gamma_{\sigma(m)}(x_m)$$

on simple tensors. It is important to notice that the sign of the permutation does not show up in this formula.

Now, let us fix a field  $\mathbb{F}$  and a natural number  $n$ . We let  $A = (A_1, \dots, A_n) \in \mathcal{L}(E)^n$  be a commuting tuple of linear operators on some vector space  $E$  over  $\mathbb{F}$ . Thus, we have the relation

$$A_i A_j - A_j A_i = 0 \quad \text{for all } i, j \in \{1, \dots, n\}$$

**3.1. The Koszul complex.** We let  $\Lambda(\mathbb{F}^n)$  denote the exterior algebra over  $\mathbb{F}^n$ . The exterior product will be denoted by

$$\wedge : \Lambda(\mathbb{F}^n) \times \Lambda(\mathbb{F}^n) \rightarrow \Lambda(\mathbb{F}^n)$$

We will think of the exterior algebra as a  $\mathbb{Z}_2$ -graded vector space with grading given by even and odd exterior powers. Thus,

$$\Lambda(\mathbb{F}^n) = \Lambda_+(\mathbb{F}^n) \oplus \Lambda_-(\mathbb{F}^n)$$

where  $\Lambda_+(\mathbb{F}^n) := \bigoplus_{m=2k} \Lambda_m(\mathbb{F}^n)$  and  $\Lambda_-(\mathbb{F}^n) := \bigoplus_{m=2k+1} \Lambda_m(\mathbb{F}^n)$ .

Let  $j \in \{1, \dots, n\}$  and let us look at the inclusion

$$\iota_j : \mathbb{F}^{n-1} \rightarrow \mathbb{F}^n \quad \iota_j(\lambda_1, \dots, \lambda_{n-1}) = (\lambda_1, \dots, \lambda_{j-1}, 0, \lambda_j, \dots, \lambda_{n-1})$$

This linear map induces an even algebra homomorphism

$$\iota_j : \Lambda(\mathbb{F}^{n-1}) \rightarrow \Lambda(\mathbb{F}^n)$$

at the level of exterior algebras.

We could also look at the projection

$$\iota_j^* : \mathbb{F}^n \rightarrow \mathbb{F}^{n-1} \quad \iota_j^*(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n)$$

This linear map induces an even algebra homomorphism

$$\iota_j^* : \Lambda(\mathbb{F}^n) \rightarrow \Lambda(\mathbb{F}^{n-1})$$

at the level of exterior algebras.

Next, we let

$$\varepsilon_j := e_j \wedge \cdot : \Lambda(\mathbb{F}^n) \rightarrow \Lambda(\mathbb{F}^n)$$

denote the odd linear operator given by exterior multiplication from the left by the  $j^{\text{th}}$  standard basis vector  $e_j \in \mathbb{F}^n$ . Finally, we let

$$\varepsilon_j^* : \Lambda(\mathbb{F}^n) \rightarrow \Lambda(\mathbb{F}^n)$$

denote the odd linear operator given by interior multiplication by the  $j^{\text{th}}$  standard dual basis vector  $e_j^* \in (\mathbb{F}^n)^*$ . Thus, to be explicit we have the formula

$$\varepsilon_j^* : e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto \begin{cases} 0 & \text{for } j \notin \{i_1, \dots, i_k\} \\ (-1)^{m-1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_m}} \wedge \dots \wedge e_{i_k} & \text{for } j = i_m \end{cases}$$

on the standard basis vectors of the exterior algebra  $\Lambda(\mathbb{F}^n)$ .

Now, let us look at the commuting tuple of linear operators  $A = (A_1, \dots, A_n) \in \mathcal{L}(E)^n$ . By the *Koszul complex of  $A$*  we will then understand the  $\mathbb{Z}_2$ -graded chain complex given by the following data:

- (1) The  $\mathbb{Z}_2$ -graded vector space

$$K(A) := E \otimes_{\mathbb{F}} \Lambda(\mathbb{F}^n)$$

Here the grading is given by the even and odd components

$$K_+(A) := E \otimes_{\mathbb{F}} \Lambda_+(\mathbb{F}^n) \quad \text{and} \quad K_-(A) := E \otimes_{\mathbb{F}} \Lambda_-(\mathbb{F}^n)$$

- (2) The differential is given by

$$d^A := \sum_{j=1}^n A_j \otimes \varepsilon_j^* : K(A) \rightarrow K(A)$$

Notice that  $(d^A)^2 = 0$  since the linear operators  $A_1, \dots, A_n \in \mathcal{L}(E)$  commute whereas the interior multiplication operators anti-commute. We will use the notation  $K(A)$  for the Koszul complex and the notation  $H(A) = H_+(A) \oplus H_-(A)$  for the  $\mathbb{Z}_2$ -graded homology of  $K(A)$ .

The Koszul complex will play a central role in the present paper.

**3.2. Fredholm theory in several variables.** The Koszul complex can be used to generalize the Fredholm theory of linear operators on vector spaces to a multivariable setting.

**Definition 3.2.1.** *We will say that the commuting tuple  $A = (A_1, \dots, A_n)$  is Fredholm when the homology of the associated Koszul complex is finite dimensional. In this case we define the Fredholm index of  $A$  as minus the Euler characteristic of the Koszul complex. Thus,*

$$\text{Ind}(A) := -\chi(K(A)) = \dim(H_-(A)) - \dim(H_+(A))$$

*Clearly, the index is an integer.*

The Fredholm index has a couple of good algebraic properties which we will now state. First of all the Fredholm index is symmetric. Let  $\sigma \in \Sigma_n$  be a permutation and let  $\sigma(A)$  denote the commuting tuple

$$\sigma(A) := (A_{\sigma(1)}, \dots, A_{\sigma(n)})$$

**Theorem 3.2.2.** *The commuting tuple  $\sigma(A)$  is Fredholm if and only if the commuting tuple  $A$  is Fredholm. In this case the two indices coincide,*

$$\text{Ind}(A) = \text{Ind}(\sigma(A))$$

*Notice that the sign of the permutation does not show up in this formula.*

*Proof.* This follows immediately by noting that the two Koszul complexes are isomorphic by the even chain map

$$1 \otimes \sigma^{-1} : K(A) \rightarrow K(\sigma(A))$$

Here  $\sigma^{-1} : \Lambda(\mathbb{F}^n) \rightarrow \Lambda(\mathbb{F}^n)$  is the even algebra isomorphism defined on generators by

$$\sigma^{-1} : e_j \mapsto e_{\sigma^{-1}(j)}$$

□

Secondly, the Fredholm index satisfies a triviality property.

**Theorem 3.2.3.** *Suppose that  $A$  is Fredholm and let  $A_{n+1} \in \mathcal{L}(E)$  be an extra linear operator which commutes with all the operators  $A_1, \dots, A_n \in \mathcal{L}(E)$ . Then the commuting tuple  $A \cup A_{n+1} := (A_1, \dots, A_{n+1})$  is Fredholm and the index is trivial,*

$$\text{Ind}(A \cup A_{n+1}) = 0$$

*Proof.* This will turn out to be a consequence of Theorem 3.3.1 and general properties of the Euler characteristic. □

Finally, the Fredholm index is additive in each variable. Let  $B = (B_1, A_2, \dots, A_n)$  be a commuting tuple of linear operators which only differ from  $A$  in the first coordinate. We then define the product  $A \cdot B$  as the commuting tuple

$$A \cdot B := (A_1 \cdot B_1, A_2, \dots, A_n)$$

**Theorem 3.2.4.** *If two of the three commuting tuples  $A, B$  and  $A \cdot B$  are Fredholm then the third one is also Fredholm. In this case the indices satisfy the additivity relation*

$$\text{Ind}(A \cdot B) = \text{Ind}(A) + \text{Ind}(B)$$

*Proof.* This will turn out to be a consequence of Lemma 5.2.1 and general properties of the Euler characteristic. □

Remark that a similar additivity result holds for the other entries. This follows immediately from Theorem 3.2.4 by the symmetry property of the Fredholm index, see Theorem 3.2.2.

The theorems stated above are only modifications of known results. Let us give some appropriate references.

A result which is similar to Theorem 3.2.2 has been proved by R. Curto [7, Proposition 9.6]. Next, a result which is similar to Theorem 3.2.3 has also been proved by R. Curto [7, §8(i)]. However, the proof of the triviality result relies on the homotopy invariance of the Fredholm index in the setting of Hilbert spaces. The proof can therefore not be directly translated to the algebraic setup we are considering in the present paper. Finally, a result which is similar



to Theorem 3.2.4 has been proved by X. Fang [14, Proposition 1]. However, it requires the additional assumption that  $A_1$  and  $B_1$  commute. Our Theorem 3.2.4 can thus be regarded as a slight improvement of [14, Proposition 1].

**Remark 3.2.5.** *When  $E$  is a Hilbert space and the commuting operators  $A_1, \dots, A_n \in \mathcal{L}(E)$  are bounded, there is an interpretation of the Fredholmness condition in terms of the joint essential spectrum of the commuting tuple  $A = (A_1, \dots, A_n)$ , see [7, §6 Theorem 2]. It follows that the Fredholmness condition is well behaved under the analytic functional calculus as developed by J. Taylor in [18, 19]. Furthermore, in this setup, the Fredholm index is invariant under homotopies and compact perturbations, see [7, §7 Theorem 3]. We will apply some of these more analytic results in the examples even though the main focus will be on the purely algebraic aspects of the theory.*

The joint torsion, which we will soon define, can be understood as a multiplicative analog of the Fredholm tuple index.

**3.3. The joint torsion transition numbers.** Let  $j \in \{1, \dots, n\}$ . We will use the notation

$$j(A) := (A_1, \dots, \widehat{A_j}, \dots, A_n)$$

for the commuting tuple obtained from the commuting tuple  $A = (A_1, \dots, A_n)$  by removing the entry in position  $j$ .

The linear operator  $A_j \in \mathcal{L}(E)$  then acts on the Koszul complex of  $j(A)$  by means of the chain map

$$A_j := A_j \otimes 1 : K(j(A)) \rightarrow K(j(A))$$

Furthermore, the Koszul complex of  $j(A)$  can be regarded as a sub-complex of the Koszul complex of  $A$  by means of the chain map

$$(3.2) \quad \iota_j := 1 \otimes \iota_j : K(j(A)) \rightarrow K(A)$$

Finally, the shifted Koszul complex of  $j(A)$  can be regarded as a quotient of the Koszul complex of  $A$  by means of the chain map

$$(3.3) \quad \iota_j^* \varepsilon_j^* := 1 \otimes \iota_j^* \varepsilon_j^* : K(A) \rightarrow K(j(A))[1]$$

Here the notation  $X[1]$  refers to the  $\mathbb{Z}_2$ -graded chain complex obtained from some  $\mathbb{Z}_2$ -graded chain complex  $X$  by reversing the grading and changing the sign of the differential.

In particular, we have an odd triangle of chain complexes in the sense of Definition 2.3.1,

$$X_j^A : K(j(A)) \xrightarrow{A_j} K(j(A))[1] \xrightarrow{\iota_j} K(A) \xrightarrow{\iota_j^* \varepsilon_j^*} K(j(A))$$

We would like to prove that the triangle  $X_j^A$  is homotopy exact in the sense of Definition 2.3.2. To this end we define the opposite triangle

$$(X_j^A)^\dagger : K(j(A)) \xleftarrow{0} K(j(A))[1] \xleftarrow{\iota_j^*} K(A) \xleftarrow{\varepsilon_j \iota_j} K(j(A))$$

Here the odd linear maps included are abbreviated in the same way as in (3.2) and (3.3).

**Theorem 3.3.1.** *The triangle  $X_j^A$  is homotopy exact with homotopy given by the opposite triangle  $(X_j^A)^\dagger$ .*

*Proof.* Let us start by checking the identities

$$0 = \iota_j^* \varepsilon_j^* \iota_j \quad -d^{j(A)} \iota_j^* + \iota_j^* d^A = A_j \iota_j^* \varepsilon_j^* \quad d^A \varepsilon_j \iota_j + \varepsilon_j \iota_j d^{j(A)} = \iota_j A_j$$

The first identity is immediate. The identity in the middle can be proved by an application of the relations

$$\varepsilon_i^* \iota_j^* = \begin{cases} \iota_j^* \varepsilon_i^* & \text{for } i < j \\ \iota_j^* \varepsilon_{i+1}^* & \text{for } i \geq j \end{cases}$$

between the projections and the interior multiplication operators. The last identity can be proved by an application of the relations

$$\varepsilon_i^* \varepsilon_j = \begin{cases} -\varepsilon_j \varepsilon_i^* & \text{for } i \neq j \\ 1 - \varepsilon_j \varepsilon_j^* & \text{for } i = j \end{cases}$$

between the exterior and interior multiplication operators. Furthermore, it is convenient to use that  $\iota_j : K(j(A)) \rightarrow K(A)$  is a chain map.

Next, we should prove the identities

$$\iota_j^* \iota_j = 1 \quad \iota_j \iota_j^* + \varepsilon_j \iota_j \iota_j^* \varepsilon_j^* = 1 \quad \iota_j^* \varepsilon_j^* \varepsilon_j \iota_j = 1$$

at the level of homology. However, it is not hard to see that they are actually valid at the level of chain complexes.  $\square$

As a consequence of Theorem 3.3.1 and Lemma 2.3.3 we get a six term exact sequence of homology groups

$$\begin{array}{ccccc} H_+(j(A)) & \xrightarrow{A_j} & H_+(j(A)) & \xrightarrow{\iota_j} & H_+(A) \\ \iota_j^* \varepsilon_j^* \uparrow & & & & \downarrow \iota_j^* \varepsilon_j^* \\ H_-(A) & \xleftarrow{\iota_j} & H_-(j(A)) & \xleftarrow{A_j} & H_-(j(A)) \end{array}$$

Suppose now that the commuting tuple  $j(A)$  is Fredholm. By definition, this means that the  $\mathbb{Z}_2$ -graded homology group  $H(j(A))$  is finite dimensional. It then follows from our six term exact sequence that the homology group  $H(A)$  is finite dimensional as well. Or in other words, the commuting tuple  $A$  is Fredholm. Furthermore, the index of  $A$  is trivial,  $\text{Ind}(A) = 0$ . Notice that this gives a proof of Theorem 3.2.3.

However, we can also use our six term exact sequence of homology groups to construct a torsion isomorphism

$$T(X_j^A) : \det(H_+(X_j^A)) \rightarrow \det(H_-(X_j^A))$$

See Definition 2.3.4. Let us recall that the finite dimensional vector spaces  $H_+(X_j^A)$  and  $H_-(X_j^A)$  are given by

$$\begin{aligned} H_+(X_j^A) &= H_+(j(A)) \oplus H_-(j(A)) \oplus H_+(A) \quad \text{and} \\ H_-(X_j^A) &= H_-(j(A)) \oplus H_+(j(A)) \oplus H_-(A) \end{aligned}$$

See (2.1). Now, the appearance of the homology group  $H(j(A))$  as a direct summand in both of the above vector spaces allows us to factor out this component from our torsion isomorphism.

Indeed, using an isomorphism which is similar to the isomorphism in (3.1), we get that

$$(3.4) \quad \text{Hom}\left(\det(H_+(X_j^A)), \det(H_-(X_j^A))\right) \cong \text{Hom}\left(\det(H_+(A)), \det(H_-(A))\right)$$

We will use the notation

$$T_j(A) \in \text{Hom}\left(\det(H_+(A)), \det(H_-(A))\right) - \{0\}$$

for the isomorphism obtained from the torsion isomorphism

$$T(X_j^A) \in \text{Hom}\left(\det(H_+(X_j^A)), \det(H_-(X_j^A))\right) - \{0\}$$

by applying the isomorphism of (3.4).

Now, let  $i \in \{1, \dots, n\}$  and suppose that the commuting tuple

$$i(A) = (A_1, \dots, \widehat{A_i}, \dots, A_n)$$

is Fredholm as well. Applying the above constructions we obtain another isomorphism

$$T_i(A) : \det(H_+(A)) \rightarrow \det(H_-(A))$$

We can thus form the composition

$$T_j(A)^{-1} \circ T_i(A) \in \text{Aut}(\det(H_+(A)))$$

which is an automorphism of the one-dimensional vector space  $\det(H_+(A))$ . Or in other words, we can associate an invertible number to the pair  $(i, j)$  and the commuting tuple  $A$ .

**Definition 3.3.2.** *By the joint torsion transition number in position  $(i, j)$  of the commuting tuple  $A$  we will understand the non-zero number*

$$\tau_{i,j}(A) \in \mathbb{F}^*$$

*obtained from the automorphism*

$$(-1)^{\mu_j(A) + \mu_i(A)} T_j(A)^{-1} \circ T_i(A) \in \text{Aut}(\det(H_+(A))) \cong \mathbb{F}^*$$

*Here the exponent for the sign is given by products of dimensions of homology groups*

$$(3.5) \quad \mu_k(A) := \dim(H_+(k(A))) \cdot \dim(H_-(k(A))) \in \mathbb{N} \cup \{0\} \quad k = i, j$$

It should be remarked that in the case  $n = 2$ , that is, when the commuting tuple  $A$  consists of two commuting Fredholm operators we recover the Carey-Pincus joint torsion. Thus, we have the identity

$$(3.6) \quad \tau_{1,2}(A_1, A_2) = \tau(A_1, A_2)$$

Here  $\tau(A_1, A_2) \in \mathbb{F}^*$  denotes the joint torsion defined by R. Carey and J. Pincus in [4]. This relation can be verified by carefully keeping track of the signs in the two different definitions.

The use of the word "transition" in Definition 3.3.2 is justified by the next lemma.

**Lemma 3.3.3.** *Suppose that the commuting tuples  $i(A)$ ,  $j(A)$  and  $k(A)$  are Fredholm for three numbers  $i, j, k \in \{1, \dots, n\}$ . Then the joint torsion transition numbers satisfy the transition identities*

$$\tau_{i,j}^{-1}(A) = \tau_{j,i}(A) \quad \text{and} \quad \tau_{i,j}(A) \cdot \tau_{j,k}(A) = \tau_{i,k}(A)$$

*Proof.* This is a straight forward verification. Indeed, the transition numbers are defined as quotients of determinants.  $\square$

**Remark 3.3.4.** *The result of Lemma 3.3.3 indicates that it should be possible to define a joint torsion line bundle by gluing together appropriate joint torsion transition functions. For example, in the Hilbert space setup each of the joint torsion transition number  $\tau_{i,j}(A)$  can be extended to a transition function over a suitable open subset  $U_i \cap U_j$  of  $\mathbb{C}^n$ . This open subset is related to the joint essential spectrum of the bounded operators involved. This would give rise to an interesting joint torsion line bundle which could be the subject of future research. The aim of the present text is however to investigate the properties of the above joint torsion transition numbers.*

Let us end this section by studying the behaviour of the joint torsion transition numbers under permutations. Thus, let  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  be a permutation. As in Section 3.2 we can form the commuting tuple

$$\sigma(A) = (A_{\sigma(1)}, \dots, A_{\sigma(n)})$$

It follows from Theorem 3.2.2 that the commuting tuples

$$\sigma^{-1}(i)(\sigma(A)) \quad \sigma^{-1}(j)(\sigma(A))$$

are Fredholm. Recall in this respect that  $i(A)$  and  $j(A)$  are assumed to be Fredholm. In particular, we can make sense of the joint torsion transition number in position  $(\sigma^{-1}(i), \sigma^{-1}(j))$ ,

$$\tau_{\sigma^{-1}(i), \sigma^{-1}(j)}(\sigma(A)) \in \mathbb{F}^*$$

of the permuted tuple  $\sigma(A)$ . Before proving a symmetry property for the joint torsion we introduce some extra maps.

Let  $k \in \{1, \dots, n\}$ . We define the shift isomorphism

$$\iota_k : \{1, \dots, n-1\} \rightarrow \{1, \dots, n\} - \{k\}$$

by the formula

$$\iota_k : m \mapsto \begin{cases} m & \text{for } m < k \\ m+1 & \text{for } m \geq k \end{cases}$$

Furthermore, we define the permutation  $k(\sigma) \in \Sigma_{n-1}$  by the formula

$$k(\sigma) := \iota_{\sigma(k)}^{-1} \circ \sigma \circ \iota_k : \{1, \dots, n-1\} \rightarrow \{1, \dots, n-1\}$$

Notice that the image of  $\sigma \circ \iota_k$  is contained in the set  $\{1, \dots, n\} - \{\sigma(k)\}$ . We then have the identity

$$(3.7) \quad k(\sigma(A)) = k(\sigma)(\sigma(k)(A))$$

of commuting tuples.

**Theorem 3.3.5.** *The joint torsion transition number in position  $(i, j)$  of the commuting tuple  $A$  coincides with the joint torsion transition number in position  $(\sigma^{-1}(i), \sigma^{-1}(j))$  of the permuted tuple  $\sigma(A)$ . Thus, in formulas we have that*

$$\tau_{i,j}(A) = \tau_{\sigma^{-1}(i), \sigma^{-1}(j)}(\sigma(A))$$

*Proof.* It follows from (3.7) that we have an isomorphism

$$k(\sigma)^{-1} : K(\sigma(k)(A)) \rightarrow K(k(\sigma(A)))$$

of Koszul complexes. See the proof of Theorem 3.2.2. This chain map fits in a commutative diagram

$$\begin{array}{ccccccc} K(\sigma(k)(A)) & \xrightarrow{A_{\sigma(k)}} & K(\sigma(k)(A))[1] & \xrightarrow{\iota_{\sigma(k)}} & K(A) & \xrightarrow{\iota_{\sigma(k)}^* \varepsilon_{\sigma(k)}^*} & K(\sigma(k)(A)) \\ k(\sigma)^{-1} \downarrow & & k(\sigma)^{-1} \downarrow & & \sigma^{-1} \downarrow & & k(\sigma)^{-1} \downarrow \\ K(k(\sigma(A))) & \xrightarrow{A_{\sigma(k)}} & K(k(\sigma(A)))[1] & \xrightarrow{\iota_k} & K(\sigma(A)) & \xrightarrow{\iota_k^* \varepsilon_k^*} & K(k(\sigma(A))) \end{array}$$

where the rows are odd homotopy exact triangles and the columns are even isomorphisms of chain complexes. In particular, we get the identities

$$\begin{aligned} T(X_i^A) &= \det(H_-(\sigma^{-1}))^{-1} \circ T(X_{\sigma^{-1}(i)}^{\sigma(A)}) \circ \det(H_+(\sigma^{-1})) \\ T(X_j^A) &= \det(H_-(\sigma^{-1}))^{-1} \circ T(X_{\sigma^{-1}(j)}^{\sigma(A)}) \circ \det(H_+(\sigma^{-1})) \end{aligned}$$

of torsion isomorphisms. Here the even isomorphisms

$$H(\sigma^{-1}) : H(X_{\sigma(k)}^A) \rightarrow H(X_k^{\sigma(A)}) \quad \sigma(k) = i, j$$

are induced by the columns of the above commutative diagram. The desired result is now a consequence of these observations and some basic properties of determinants.  $\square$

**3.4. Example : Lefschetz numbers.** Let  $i, j \in \{1, \dots, n\}$  be two numbers and suppose that the commuting tuples

$$i(A) = (A_1, \dots, \widehat{A_i}, \dots, A_n) \quad \text{and} \quad j(A) = (A_1, \dots, \widehat{A_j}, \dots, A_n)$$

are Fredholm. Furthermore, suppose that the Koszul homology of  $A$  is trivial, thus  $H(A) = \{0\}$ .

The Fredholm condition on  $i(A)$  and  $j(A)$  ensures us that the joint torsion transition number  $\tau_{i,j}(A) \in \mathbb{F}^*$  is well-defined. Furthermore, the vanishing condition on the Koszul homology of  $A$  entails that the even chain maps

$$A_i : K(i(A)) \rightarrow K(i(A)) \quad \text{and} \quad A_j : K(j(A)) \rightarrow K(j(A))$$

induces isomorphisms at the level of homology. This is a consequence of Theorem 3.3.1. The next theorem gives an expression for the joint torsion transition number in terms of quotients of determinants of these induced isomorphisms on homology.

**Theorem 3.4.1.** *The joint torsion transition number  $\tau_{i,j}(A) \in \mathbb{F}^*$  coincides with the product of quotients of determinants*

$$\tau_{i,j}(A) = \left( \frac{\det(A_i|_{H_+(i(A))}}{\det(A_i|_{H_-(i(A))})} \right) \cdot \left( \frac{\det(A_j|_{H_-(j(A))}}{\det(A_j|_{H_+(j(A))})} \right)$$

Here the notation

$$A_k|_{H_{\pm}(k(A))} : H_{\pm}(k(A)) \rightarrow H_{\pm}(k(A)) \quad k = i, j$$

refers to the restriction of  $A_k$  to the positive or negative part of the Koszul homology of the commuting tuple  $k(A)$ .

*Proof.* The  $\mathbb{Z}_2$ -graded vector space  $H(X_i^A)$  associated with the odd homotopy exact triangle  $X_i^A$  is given by the components

$$H_+(X_i^A) = H(i(A)) \quad \text{and} \quad H_-(X_i^A) = H(i(A))[1]$$

Here the notation "[1]" refers to the operation of taking the opposite grading. Furthermore, the odd exact endomorphism

$$H(\alpha_i) \in \text{End}_-(H(X_i^A))$$

associated with the odd homotopy exact triangle  $X_i^A$  is given by the matrices

$$\begin{aligned} H_+(\alpha_i) &= \begin{pmatrix} 0 & 0 \\ A_i & 0 \end{pmatrix} : H(i(A)) \rightarrow H(i(A))[1] \quad \text{and} \\ H_-(\alpha_-) &= \begin{pmatrix} 0 & 0 \\ A_i & 0 \end{pmatrix} : H(i(A))[1] \rightarrow H(i(A)) \end{aligned}$$

See (2.2). The torsion isomorphism of the odd homotopy exact triangle  $X_i^A$  is therefore given by the determinant

$$T(X_i^A) = \det \begin{pmatrix} 0 & A_i^{-1} \\ A_i & 0 \end{pmatrix} : \det(H(i(A))) \rightarrow \det(H(i(A))[1])$$

By an application of the isomorphisms in (3.4) we get that  $T(X_i^A)$  identifies with the quotient of determinants

$$T_i(A) = (-1)^{\mu_i(A)} \frac{\det(A_i|_{H_+(i(A))})}{\det(A_i|_{H_-(i(A))})} \in \mathbb{F}^*$$

Here the exponent  $\mu_i(A) \in \mathbb{N} \cup \{0\}$  can be found in Definition 3.3.2. The result of the theorem now follows by noting that the same calculations can be applied when  $i$  is replaced by  $j$ .  $\square$

The product of quotients of determinants obtained in Theorem 3.4.1 is referred to by R. Carey and J. Pincus as a "Lefschetz number". See [4, §4 p. 289]. In the case of a pair of commuting Fredholm operators with vanishing Koszul homology their associated Lefschetz number has been studied in [5]. In particular, the relation between these numbers and the second algebraic  $K$ -group was clarified. One of the motivations for introducing a joint torsion invariant is to generalize the notion of a Lefschetz number to the more general setup where the vanishing condition on Koszul homology is removed. See [6, §1 p. 128]. The result of Theorem 3.4.1 therefore gives some justification for our definition of a joint torsion transition number.

**3.5. Example : Toeplitz operators over the polydisc.** Let  $n \in \mathbb{N}$ , let  $\mathbb{T}^n \subseteq \mathbb{C}^n$  be the  $n$ -dimensional torus and let  $\mathbb{D}^n \subseteq \mathbb{C}^n$  be the polydisc. The interior of  $\mathbb{D}^n$  will be denoted by  $U^n$ .

We let  $L^2(\mathbb{T}^n)$  denote the Hilbert space of square integrable functions on the torus. The continuous functions on the torus act by pointwise multiplication on the  $L^2$ -functions. Thus, we have an algebra homomorphism

$$m : C(\mathbb{T}^n) \rightarrow \mathcal{L}(L^2(\mathbb{T}^n)) \quad m(f)(g) = f \cdot g$$

Here  $\mathcal{L}(L^2(\mathbb{T}^n))$  denotes the bounded operators on the Hilbert space  $L^2(\mathbb{T}^n)$ .

We let  $\mathcal{A}(U^n)$  denote the polydisc algebra. Thus,  $\mathcal{A}(U^n)$  consists of the continuous functions on the polydisc  $\mathbb{D}^n$  which restrict to holomorphic functions on the interior of the poly-disc,  $U^n$ . The algebraic operations on  $\mathcal{A}(U^n)$  are the pointwise versions of sum, product and scalar multiplication. See [17].

We let  $H^2(U^n) \subseteq L^2(\mathbb{T}^n)$  denote the Hardy-space over the poly-disc. This is the smallest sub-Hilbert space of the  $L^2$ -functions generated by the continuous functions which extend to elements of the polydisc algebra. The orthogonal projection onto Hardy-space will be denoted by  $P \in \mathcal{L}(L^2(\mathbb{T}^n))$ .

**Definition 3.5.1.** *By the Toeplitz operator associated to a continuous function  $f \in C(\mathbb{T}^n)$  we will understand the restriction of its multiplication operator to Hardy space. The corresponding Toeplitz operator will be denoted by  $T_f \in \mathcal{L}(H^2(U^n))$ . Thus, by definition*

$$T_f = Pm(f)P : H^2(U^n) \rightarrow H^2(U^n)$$

We let  $z_1, \dots, z_n : \mathbb{T}^n \rightarrow \mathbb{C}$  denote the coordinate functions on the torus.

Now, let  $f \in \mathcal{A}(U^n)$  be an *invertible* element of the poly-disc algebra. By a slight abuse of notation we will also let  $f \in C(\mathbb{T}^n)$  denote the restriction of  $f$  to the  $n$ -torus. We will look at the commuting tuple of Toeplitz operators

$$T_\alpha := (T_f, T_{z_1 - \alpha_1}, \dots, T_{z_n - \alpha_n})$$

where  $\alpha = (\alpha_1, \dots, \alpha_n) \in U^n$  is an element of the interior of the poly-disc. Notice that the invertibility condition on  $f$  implies that the Koszul homology groups

$$H(T_\alpha) = \{0\} = H(i(T_\alpha)) \quad i \neq 1$$

vanish. Furthermore, it is not hard to show, that the Koszul homology of  $1(T_\alpha)$  is given by the components

$$H_+(1(T_\alpha)) = \mathcal{H} / (m(z_1 - \alpha_1)\mathcal{H} + \dots + m(z_n - \alpha_n)\mathcal{H}) \quad \text{and} \quad H_-(1(T_\alpha)) = \{0\}$$

Here  $\mathcal{H} = H^2(U^n)$  is notation for the Hardy-space over the poly-disc. The dimension of  $H_+(1(T_\alpha))$  is equal to one. One way of seeing this, is to think of the case where  $\alpha = (0, \dots, 0)$ . In this situation, we get that  $H_+(1(T_\alpha))$  is spanned by the constant function equal to one. To obtain the result in the general setting it now suffices to use the homotopy invariance of the Fredholm tuple index. See [7]. The dimension of  $H_+(1(T_\alpha))$  can also be found by a direct calculation.

The finite dimensionality of the quotient space implies that the subspace

$$m(z_1 - \alpha_1)\mathcal{H} + \dots + m(z_n - \alpha_n)\mathcal{H} \subseteq \mathcal{H}$$

is closed. See [7, §6 Theorem 2]. These observations allow us to compute the joint torsion transition numbers of our commuting tuple of Toeplitz operators.

**Theorem 3.5.2.** *Let  $1 \leq i < j \leq n$  be two numbers. The joint torsion transition numbers of the commuting tuple  $T_\alpha$  are given by*

$$\tau_{i,j}(T_\alpha) = \begin{cases} f(\alpha) & \text{for } i = 1 \\ 1 & \text{for } i \neq 1 \end{cases}$$

*Proof.* We will only consider the case where  $i = 1$ . By an application of Theorem 3.4.1 we get that the joint torsion transition number  $\tau_{1,j}(T_\alpha)$  is given by the determinant of

$$T_f : H_+(1(T_\alpha)) \rightarrow H_+(1(T_\alpha))$$

But this isomorphism coincides with the multiplication by the non-zero constant  $f(\alpha) \in \mathbb{C}^*$ . This proves the theorem.  $\square$

Remark that the above calculation yields an interesting link between the joint torsion transition numbers and the Cauchy integral formula. Indeed, we have that

$$\frac{1}{(2\pi i)^n} \oint \frac{f(z)}{(z_1 - \alpha_1) \cdots (z_n - \alpha_n)} dz = f(\alpha) = \tau_{1,j}(T_f, T_{z_1 - \alpha_1}, \dots, T_{z_n - \alpha_n})$$

Furthermore, it shows that the joint torsion transition numbers are far from being homotopy invariant: We could reproduce all the values of the invertible holomorphic function  $f$  on the interior of the polydisc.

#### 4. COMPARISON OF VERTICAL AND HORIZONTAL TORSION ISOMORPHISMS

In this section we study the basic question: When do the torsion isomorphisms of two anti-commuting odd exact endomorphisms on a finite dimensional  $\mathbb{Z}_2$ -graded vector space coincide? We shall see that this is the case when the odd exact endomorphisms come from odd homotopy exact triangles which fit together as rows and columns in a larger diagram satisfying some (anti)-commutativity conditions. Later on we will apply this comparison theorem to prove some more involved results on our joint torsion transition numbers. In the first subsection we establish the framework of odd homotopy exact bitriangles. The definitions which we give are minimal in the sense that they are tailored to handle the applications which we have in mind. In the second subsection we compare the torsion isomorphisms of general anti-commuting odd exact endomorphisms. We think that the determinants appearing in Theorem 4.2.5 are an obstruction for such a pair of torsion isomorphisms to agree in general. In the last subsection we prove that the vertical and horizontal torsion isomorphisms of an odd homotopy exact bitriangle agree. In particular, the obstruction mentioned above vanishes in this situation. The result of this section might be interpreted as the associativity of a determinant functor on a certain triangulated category. See [2, Definition 3.1] and [13, Definition 1.3.4].

**4.1. Odd homotopy exact bitriangles.** Let us consider three odd triangles of  $\mathbb{Z}_2$ -graded chain complexes,  $X_{*1}, X_{*2}$  and  $X_{*3}$ . Thus, to fix the notation we have the following picture

$$X_{*j} : X_{1j} \xrightarrow{v_{1j}} X_{2j} \xrightarrow{v_{2j}} X_{3j} \xrightarrow{v_{3j}} X_{1j}$$

for each  $j \in \{1, 2, 3\}$ . We will think of these three odd triangles as being "vertical". Let us then look at three odd triangles of chain complexes  $X_{1*}, X_{2*}$  and  $X_{3*}$ . Thus, again, to fix the notation we have the following picture

$$X_{i*} : X_{i1} \xrightarrow{h_{i1}} X_{i2} \xrightarrow{h_{i2}} X_{i3} \xrightarrow{h_{i3}} X_{i1}$$

for each  $i \in \{1, 2, 3\}$ . We will think of these three triangles as being "horizontal".



The differential of the  $\mathbb{Z}_2$ -graded chain complex  $X_{i,j}$  will be denoted by

$$d_{i,j} : X_{i,j} \rightarrow X_{i,j}$$

**Definition 4.1.1.** *We will say that the data  $(X, v, h)$  is an odd bitriangle of chain complexes when the diagrams of odd chain maps*

$$\begin{array}{ccc} X_{i,j} & \xrightarrow{h_{i,j}} & X_{i,(j+1)} \\ v_{i,j} \downarrow & & v_{i,(j+1)} \downarrow \\ X_{(i+1),j} & \xrightarrow{h_{(i+1),j}} & X_{(i+1),(j+1)} \end{array}$$

*are anti-commutative up to chain homotopy for all  $i, j \in \{1, 2, 3\}$ . Thus, we assume the existence of odd linear maps*

$$s_{i,j} : X_{i,j} \rightarrow X_{(i+1),(j+1)}$$

*such that*

$$d_{(i+1),(j+1)} s_{i,j} + s_{i,j} d_{i,j} = h_{(i+1),j} v_{i,j} + v_{i,(j+1)} h_{i,j}$$

*for all  $i, j \in \{1, 2, 3\}$ . Here we are calculating with the indices modulo three.*

Now, suppose that  $(X, v, h)$  is an odd bitriangle of chain complexes. Furthermore, suppose that each of the vertical odd triangles and each of the horizontal odd triangles are homotopy exact. See Definition 2.3.2. We will use the notation

$$t_{i,j} : X_{(i+1),j} \rightarrow X_{i,j} \quad \text{and} \quad r_{i,j} : X_{i,(j+1)} \rightarrow X_{i,j}$$

for the corresponding vertical and horizontal homotopies.

We will often suppress the bi-indices of the linear maps involved.

**Lemma 4.1.2.** *The even linear maps*

$$vs + sv + th + ht : X_{i,j} \rightarrow X_{(i-1),(j+1)} \quad \text{and} \quad hs + sh + rv + vr : X_{i,j} \rightarrow X_{(i+1),(j-1)}$$

*are even chain maps for all  $i, j \in \{1, 2, 3\}$ .*

*Proof.* We will only consider the first of the two even maps, since the other case is completely similar. The chain map property follows from the calculation,

$$\begin{aligned} d(vs + sv + th + ht) &= -vds + (vh + hv - sd)v + (v^2 - td)h - hdt \\ &= -v(vh + hv - sd) + (vh + hv)v + svd + v^2h + thd - h(v^2 - td) \\ &= (vs + sv + th + ht)d \end{aligned}$$

□

We will now introduce the notion of homotopy exactness for odd bitriangles.

**Definition 4.1.3.** *We say that the odd bitriangle  $(X, v, h)$  with homotopy exact rows and columns is homotopy exact when the even chain maps*

$$vs + sv + th + ht : X_{i,j} \rightarrow X_{i-1,j+1} \quad \text{and} \quad hs + sh + rv + vr : X_{i,j} \rightarrow X_{i+1,j-1}$$

*vanish at the level of homology for each  $i, j \in \{1, 2, 3\}$ .*

Suppose that  $(X, v, h)$  is homotopy exact and that the homology of the  $\mathbb{Z}_2$ -graded chain complex  $X_{i,j}$  is finite dimensional for each  $i, j \in \{1, 2, 3\}$ . We thus have three "horizontal" torsion isomorphisms

$$T(X_{i*}) : \det(H_+(X_{i*})) \rightarrow \det(H_-(X_{i*})) \quad i \in \{1, 2, 3\}$$

and three "vertical" torsion isomorphisms

$$T(X_{*j}) : \det(H_+(X_{*j})) \rightarrow \det(H_-(X_{*j})) \quad j \in \{1, 2, 3\}$$

It is the goal of the next sections to compare these isomorphisms. Later on we will use this comparison result to prove some algebraic properties of the joint torsion transition numbers. The proof of these algebraic properties is the main achievement of this paper. We will begin with an analysis of the general  $\mathbb{Z}_2$ -graded situation.

**4.2. Comparison of odd exact endomorphisms.** Let us consider a  $\mathbb{Z}_2$ -graded vector space  $V = V_+ \oplus V_-$  of finite dimension. We suppose that  $V$  comes equipped with two odd exact endomorphisms

$$v : V \rightarrow V \quad \text{and} \quad h : V \rightarrow V$$

Thus, we have two torsion isomorphisms

$$\begin{aligned} T(v) &= \det(v_+ + v_-^\dagger) : \det(V_+) \rightarrow \det(V_-) \quad \text{and} \\ T(h) &= \det(h_+ + h_-^\dagger) : \det(V_+) \rightarrow \det(V_-) \end{aligned}$$

between the same one-dimensional vector spaces. These two torsion isomorphisms can then be compared by looking at the invertible number

$$(4.1) \quad T(h)^{-1} \circ T(v) = \det(h_+^\dagger + h_-) \circ \det(v_+ + v_-^\dagger) \in \mathbb{F}^*$$

This quantity will be the subject of our attention in this section and we will refer to it as the *comparison number*.

Let us assume that  $v$  and  $h$  anti-commute. Thus, we have the identity

$$vh + hv = 0 : V \rightarrow V$$

of linear maps.

We will use the notation

$$K^v := \text{Ker}(v) \quad K^h := \text{Ker}(h) \quad K^{vh} := \text{Ker}(vh) = \text{Ker}(hv)$$

for the various kernels. Furthermore, we will use the notation

$$I := \text{Im}(vh) = \text{Im}(hv)$$

for the image of the composition of the odd exact endomorphisms. Notice that all these subspaces are  $\mathbb{Z}_2$ -graded by the induced grading.

The next lemma follows from the anti-commutativity and exactness assumptions on our odd endomorphisms. We will use it to define a relevant algebraic decomposition of our  $\mathbb{Z}_2$ -graded vector space.

**Lemma 4.2.1.** *The odd exact endomorphisms  $v, h \in \text{End}_-(V)$  induce odd isomorphisms*

$$v \text{ and } h : K^{vh}/(K^v + K^h) \rightarrow (K^v \cap K^h)/I$$

*and odd isomorphisms*

$$v : V/K^{vh} \rightarrow K^v/(K^v \cap K^h) \text{ and } h : V/K^{vh} \rightarrow K^h/(K^v \cap K^h)$$

Let us choose subspaces  $R_+ \subseteq K_+^{vh}$  and  $R_- \subseteq K_-^{vh}$  such that the quotient maps

$$R_+ \rightarrow K_+^{vh}/(K_+^v + K_+^h) \text{ and } R_- \rightarrow K_-^{vh}/(K_-^v + K_-^h)$$

become isomorphisms. We let  $R := R_+ \oplus R_- \subseteq V$  denote the  $\mathbb{Z}_2$ -graded subspace given by these two components.

Likewise, let us choose subspaces  $S_+ \subseteq V_+$  and  $S_- \subseteq V_-$  such that the quotient maps

$$S_+ \rightarrow V_+/K_+^{vh} \text{ and } S_- \rightarrow V_-/K_-^{vh}$$

become isomorphisms. We let  $S := S_+ \oplus S_- \subseteq V$  denote the  $\mathbb{Z}_2$ -graded subspace given by these two components.

We will then use the notation

$$\begin{aligned} C^v &:= \text{Im}(v|_R) \text{ and } C^h := \text{Im}(h|_R) \\ L &:= \text{Im}(v|_S) \text{ and } M := \text{Im}(h|_S) \end{aligned}$$

for the various images associated with these subspaces. Notice that all these images are  $\mathbb{Z}_2$ -graded by the induced grading coming from  $V$ . We then have two different algebraic decompositions of the vector space  $V$ .

**Lemma 4.2.2.** *The vector space sum induces even isomorphisms*

$$\begin{aligned} I \oplus C^v \oplus L \oplus M \oplus R \oplus S &\cong V \quad \text{and} \\ I \oplus C^h \oplus L \oplus M \oplus R \oplus S &\cong V \end{aligned}$$

*of  $\mathbb{Z}_2$ -graded vector spaces.*

*Proof.* This follows immediately from the definitions of the subspaces and Lemma 4.2.1.  $\square$

We will use the algebraic decompositions of Lemma 4.2.2 to define convenient pseudo-inverses

$$h_+^\dagger : V_- \rightarrow V_+ \text{ and } v_-^\dagger : V_+ \rightarrow V_-$$

Indeed, we can choose the pseudo-inverse  $h_+^\dagger$  such that

$$\text{Ker}(h_+^\dagger) = L_- + R_- + S_- \text{ and } \text{Im}(h_+^\dagger) = L_+ + R_+ + S_+$$

This pseudo-inverse is given explicitly on  $\text{Im}(h_+) = I_- + C_-^h + M_-$  by the formula

$$h_+^\dagger(v_+h_-(x_1) + h_+(x_2) + h_+(x_3)) = -v_-(x_1) + x_2 + x_3 \quad x_1 \in S_-, x_2 \in R_+, x_3 \in S_+$$

Likewise, we can choose the pseudo-inverse  $v_-^\dagger$  such that

$$\text{Ker}(v_-^\dagger) = M_+ + R_+ + S_+ \text{ and } \text{Im}(v_-^\dagger) = M_- + R_- + S_-$$

This pseudo-inverse is given explicitly on  $\text{Im}(v_-) = I_+ + C_+^v + L_+$  by the formula

$$v_-^\dagger(v_-h_+(x_1) + v_-(x_2) + v_-(x_3)) = h_+(x_1) + x_2 + x_3 \quad x_1 \in S_+, x_2 \in R_-, x_3 \in S_-$$

These explicit choices of pseudo-inverses will allow us to find a simpler formula for the comparison number (4.1). In the next lemma we analyze the behaviour of the isomorphism

$$(h_- + h_+^\dagger) \circ (v_+ + v_-^\dagger) : V_+ \rightarrow V_+$$

on two invariant subspaces.

**Lemma 4.2.3.** *The subspaces  $I_+ + S_+ \subseteq V_+$  and  $L_+ + M_+ \subseteq V_+$  are invariant under the isomorphism*

$$(h_- + h_+^\dagger) \circ (v_+ + v_-^\dagger) : V_+ \rightarrow V_+$$

*Furthermore, the determinants of the restrictions*

$$\begin{aligned} \det((h_- + h_+^\dagger)(v_+ + v_-^\dagger)|_{I_+ + S_+}) &= 1 \quad \text{and} \\ \det((h_- + h_+^\dagger)(v_+ + v_-^\dagger)|_{L_+ + M_+}) &= 1 \end{aligned}$$

*are both trivial.*

*Proof.* Let us look at an arbitrary element

$$v_-h_+(x) + y \in I_+ + S_+ \quad x, y \in S_+$$

We can then calculate as follows

$$(h_- + h_+^\dagger)(v_+ + v_-^\dagger)((v_-h_+)(x) + y) = (h_- + h_+^\dagger)(h_+(x) + v_+(y)) = x + (h_-v_+)(y)$$

This proves that  $I_+ + S_+$  is an invariant subspace for the automorphism  $(h_- + h_+^\dagger)(v_+ + v_-^\dagger) \in \text{Aut}(V_+)$ . Furthermore, we see that the restriction of this automorphism to the invariant subspace in question can be written on the matrix form

$$(h_- + h_+^\dagger)(v_+ + v_-^\dagger) = \begin{pmatrix} 0 & -f \\ f^{-1} & 0 \end{pmatrix} : I_+ \oplus S_+ \rightarrow I_+ \oplus S_+$$

Here the letter  $f$  refers to the isomorphism

$$f := v_-h_+ : S_+ \rightarrow I_+$$

In particular, the determinant of the restriction to this subspace is trivial. This proves the first part of the lemma.

For the second part, we look at an arbitrary element

$$v_-(x) + h_-(y) \in L_+ + M_+ \quad x, y \in S_-$$

in the subspace  $L_+ + M_+$ . We can then calculate as follows

$$(h_- + h_+^\dagger)(v_+ + v_-^\dagger)(v_-(x) + h_-(y)) = (h_- + h_+^\dagger)(x + (v_+h_-)(y)) = h_-(x) - v_-(y)$$

This proves that the subspace  $L_+ + M_+ \subseteq V_+$  is invariant under the automorphism  $(h_- + h_+^\dagger)(v_+ + v_-^\dagger) \in \text{Aut}(V_+)$ . Furthermore, we see that the restriction can be written on the matrix form

$$(h_- + h_+^\dagger)(v_+ + v_-^\dagger) = \begin{pmatrix} 0 & -g \\ g^{-1} & 0 \end{pmatrix} : L_+ \oplus M_+ \rightarrow L_+ \oplus M_+$$

Here  $g : M_+ \rightarrow L_+$  is the isomorphism given by

$$g : h_-(x) \mapsto v_-(x) \quad x \in S_-$$

In particular, the determinant of the restriction to this subspace is trivial. These observations prove the second part of the lemma.  $\square$

Let us use notation  $P_R : V \rightarrow V$  for the unique idempotent with

$$\text{Im}(P_R) = R \quad \text{and} \quad \text{Ker}(P_R) = K^v + K^h + S$$

Furthermore, we will use the notation  $P_{C^v} : V \rightarrow V$  for the unique idempotent with

$$\text{Im}(P_{C^v}) = C^v \quad \text{and} \quad \text{Ker}(P_{C^v}) = I + L + M + R + S$$

**Lemma 4.2.4.** *The comparison number is given by the product of determinants*

$$T(h)^{-1} \circ T(v) = \det(P_R h_+^\dagger v_+|_{R_+}) \cdot \det(P_{C^v} h_- v_-^\dagger|_{C_+^v}) \in \mathbb{F}^*$$

*of restrictions to the subspaces  $R_+$  and  $C_+^v$ .*

*Proof.* We consider the restrictions

$$\begin{aligned} (h_+^\dagger + h_-)(v_+ + v_-^\dagger)|_{R_+} &= h_+^\dagger v_+|_{R_+} : R_+ \rightarrow V_+ \quad \text{and} \\ (h_+^\dagger + h_-)(v_+ + v_-^\dagger)|_{C_+^v} &= h_- v_-^\dagger|_{C_+^v} : C_+^v \rightarrow V_+ \end{aligned}$$

It can be verified that we have the inclusions

$$\text{Im}(h_+^\dagger v_+|_{R_+}) \subseteq R_+ + L_+ \quad \text{and} \quad \text{Im}(h_- v_-^\dagger|_{C_+^v}) \subseteq C_+^v + I_+$$

for the images of these restrictions. In particular, we can write the automorphism  $(h_+^\dagger + h_-)(v_+ + v_-^\dagger) \in \text{Aut}(V_+)$  as an upper triangular matrix with respect to the decomposition

$$V_+ \cong (I_+ + S_+) \oplus (L_+ + M_+) \oplus R_+ \oplus C_+^v$$

The result of the lemma is now a consequence of Lemma 4.2.3 and basic properties of determinants.  $\square$

Let us use the letters  $\xi$  and  $\eta$  for the odd isomorphisms

$$\begin{aligned} \xi &:= v : K^{vh}/(K^v + K^h) \rightarrow (K^v \cap K^h)/I \quad \text{and} \\ \eta &:= h : K^{vh}/(K^v + K^h) \rightarrow (K^v \cap K^h)/I \end{aligned}$$

induced on quotient spaces by the odd exact endomorphisms  $v$  and  $h$ . See Lemma 4.2.1.

**Theorem 4.2.5.** *The comparison number is given by the product of determinants*

$$T(h)^{-1} \circ T(v) = \det(\eta_+^{-1} \xi_+) \cdot \det(\eta_- \xi_-^{-1}) \in \mathbb{F}^*$$

*on quotient spaces.*

*Proof.* This follows immediately from Lemma 4.2.4.  $\square$

**4.3. The comparison theorem.** Let  $(X, v, h)$  be an odd homotopy exact bitriangle of chain complexes.

We define the "diagonal"  $\mathbb{Z}_2$ -graded chain complexes as the direct sums of  $\mathbb{Z}_2$ -graded chain complexes

$$(4.2) \quad \begin{aligned} D_1 &:= X_{31} \oplus X_{22} \oplus X_{13} \\ D_2 &:= X_{11} \oplus X_{32} \oplus X_{23} \\ D_3 &:= X_{21} \oplus X_{12} \oplus X_{33} \end{aligned}$$

The  $\mathbb{Z}_2$ -graded homology of the  $\mathbb{Z}_2$ -graded chain complex  $D_k$  is denoted by  $H(D_k)$  for each  $k \in \{1, 2, 3\}$ . We let  $H(X)$  denote the  $\mathbb{Z}_2$ -graded vector space given by the direct sum of homology groups

$$(4.3) \quad H(X) := H(D_1) \oplus H(D_2) \oplus H(D_3)$$

We will refer to  $H(X)$  as the *homology of the bitriangle  $X$* . Let us assume that the homology group  $H(X)$  is a finite dimensional vector space.

The vertical and horizontal odd chain maps of the odd bitriangle  $X$  induce odd endomorphisms

$$v : H(X) \rightarrow H(X) \quad \text{and} \quad h : H(X) \rightarrow H(X)$$

of the homology of  $X$ . These odd endomorphisms are exact since the columns and rows of the homotopy exact bitriangle are homotopy exact. Furthermore, they anti-commute since the odd chain maps anti-commute up to chain homotopy. We are thus in the situation of the last section. In particular, we get the identity

$$T(h)^{-1} \circ T(v) = \det(\eta_+^{-1} \xi_+) \cdot \det(\eta_- \xi_-^{-1}) \in \mathbb{F}^*$$

for the associated comparison number. See Theorem 4.2.5. Here we recall that  $\xi$  and  $\eta$  denote the odd isomorphisms

$$\xi \text{ and } \eta : K^{vh} / (K^v + K^h) \rightarrow (K^v \cap K^h) / I$$

of quotient spaces induced by the odd maps  $v, h \in \text{End}_-(H(X))$ . The purpose of this section is to show that the comparison number of an odd homotopy exact bitriangle is trivial. This will allow us to compare the torsion isomorphisms of the columns and rows of the homotopy exact bitriangle  $(X, v, h)$ . See the end of Section 4.1. We will start by obtaining a description of the space  $K^{vh} / (K^v + K^h)$  as the quotient of a homology group.

Let  $k \in \{1, 2, 3\}$ . The vertical and horizontal maps induce odd linear maps

$$v : H(D_k) \rightarrow H(D_{k+1}) \quad \text{and} \quad h : H(D_k) \rightarrow H(D_{k+1})$$

on the diagonal homology groups. These induced vertical and horizontal odd maps anti-commute and have trivial squares. We can thus define a differential

$$\delta_k : H(D_k) \rightarrow H(D_{k+1}) \quad \delta_k = v - h$$

The corresponding homology groups will be denoted by

$$H_\delta(H(D_k)) := \text{Ker}(\delta_k) / \text{Im}(\delta_{k-1})$$

Notice that these homology groups are  $\mathbb{Z}_2$ -graded vector spaces with the induced grading coming from the  $\mathbb{Z}_2$ -graded homology groups  $H(D_k)$ .

Let  $i, j \in \{1, 2, 3\}$  be two numbers with sum congruent to  $k$  modulo 3,  $i + j \equiv_3 k$ . We let

$$P_{i,j} : H(D_k) \rightarrow H(X_{i,j})$$

denote the map which is induced by the even projection onto the component  $X_{i,j}$  relative to the decomposition (4.2).

We then have a well-defined even homomorphism

$$(4.4) \quad P_{i,j} : H_\delta(H(D_k)) \rightarrow P_{i,j}(K^{vh}/(K^v + K^h))$$

of  $\mathbb{Z}_2$ -graded vector spaces.

**Lemma 4.3.1.** *The even linear map*

$$P_{i,j} : H_\delta(H(D_k)) \rightarrow P_{i,j}(K^{vh}/(K^v + K^h))$$

*is surjective.*

*Proof.* We will apply the notation of Section 4.1 for the various chain homotopies.

Suppose that  $x \in X_{i,j} = P_{i,j}D_k$  is a cycle and that we have an element  $y \in X_{(i+1),(j+1)}$  such that

$$(vh)(x) = d(y)$$

We notice that

$$(hv)(x) = (ds + sd - vh)(x) = d(sx - y)$$

Define two elements  $z, w \in D_k$  by the formulae

$$z := (th)(x) + v(y) \quad \text{and} \quad w := (rv)(x) + h(sx - y)$$

Notice that  $P_{(i-1),(j+1)}z = z$  and that  $P_{(i+1),(j-1)}w = w$ . Furthermore, from the proof of Lemma 2.3.3 we know that

$$d(z) = 0 = d(w)$$

and that we have the identities

$$(4.5) \quad v[z] = h[x] \quad \text{and} \quad h[w] = v[x]$$

in the homology group  $H(D_{k+1})$ .

We shall now see that the value of the horizontal map at the class  $[z] \in H(D_k)$  agree with the value of the vertical map at the class  $[w] \in H(D_k)$ . This follows from the calculation

$$\begin{aligned} h[z] &= [(hth)(x) + (hv)(y)] \\ &= [-h(vs + sv + ht)(x) + (sd - vh)(y)] \\ &= [(vh - sd)(sx) - (hsv)(x) - (rdt)(x) + (svh)(x) - (vh)(y)] \\ (4.6) \quad &= [(vh - sd)(sx) - (hsv)(x) - (rv^2)(x) + s(ds - hv)(x) - (vh)(y)] \\ &= [(vh)(sx - y) - (hs + rv + sh)(vx)] \\ &= [(vh)(sx - y) + (vrv)(x)] \\ &= v[w] \end{aligned}$$

Here we are using several times that  $(X, v, h)$  is an odd homotopy exact bitriangle. In particular, we use that the chain maps

$$th + ht + vs + sv \text{ and } sh + hs + rv + vr : X \rightarrow X$$

vanish at the level of homology. See Definition 4.1.3.

It follows from the identities (4.5) and (4.6) that the sum of classes

$$[x] + [z] + [w] \in \text{Ker}(\delta_k)$$

lies in the kernel of the differential  $\delta_k = v - h : H(D_k) \rightarrow H(D_{k+1})$ . This proves the lemma since  $P_{i,j}([x] + [z] + [w]) = [x]$ .  $\square$

The last lemma will allow us to give a very simple description of the even isomorphism

$$(4.7) \quad \eta^{-1}\xi : K^{vh}/(K^v + K^h) \rightarrow K^{vh}/(K^v + K^h)$$

Indeed, let us look at some element  $y \in P_{i,j}K^{vh}/(K^v + K^h)$ . It follows from Lemma 4.3.1 that we can find an element  $x \in H_\delta(H(D_k))$  with  $y = P_{i,j}x$ . In particular, we get that

$$(4.8) \quad \begin{aligned} (\eta^{-1}\xi)(P_{i,j}x) &= (\eta^{-1}P_{i+1,j}\xi)(x) \\ &= (\eta^{-1}P_{i+1,j}\eta)(x) \\ &= P_{i+1,j-1}(x) \end{aligned}$$

Thus the isomorphism  $\eta^{-1}\xi$  is nothing but a shift along the diagonal in the direction down and left.

Remark that the isomorphism in (4.7) induces even isomorphisms

$$(4.9) \quad P_{i,j}K^{vh}/(K^v + K^h) \cong P_{i-1,j+1}K^{vh}/(K^v + K^h) \cong P_{i+1,j-1}K^{vh}/(K^v + K^h)$$

of  $\mathbb{Z}_2$ -graded vector spaces.

**Lemma 4.3.2.** *The determinants*

$$\det(\eta_+^{-1}\xi_+) = 1 = \det(\eta_- \xi_-^{-1})$$

*are both trivial.*

*Proof.* We will only consider the determinant  $\det(\eta_+^{-1}\xi_+) \in \mathbb{F}^*$ . The proof is similar for the other determinant since  $\det(\eta_- \xi_-^{-1}) = \det(\xi_-^{-1}\eta_-)$ .

It follows from the description of the even isomorphism

$$\eta^{-1}\xi : K^{vh}/(K^v + K^h) \rightarrow K^{vh}/(K^v + K^h)$$

given in (4.8) that the determinant in question is a product of signs of cyclic permutations. Indeed, we have that

$$\det(\eta_+^{-1}\xi_+) = (-1)^{d(31)_+ + (d(22)_+ + d(13)_+) + d(11)_+ + (d(32)_+ + d(23)_+) + d(21)_+ + (d(12)_+ + d(33)_+)}$$

where the non-negative numbers  $d(ij)_+ \in \mathbb{N} \cup \{0\}$  are given by the dimensions of quotient spaces

$$d(ij)_+ = \dim\left(P_{ij}(K_+^{vh}/(K_+^v + K_+^h))\right)$$

The result of the lemma is now a consequence of the isomorphisms in (4.9).  $\square$



The next theorem follows from Lemma 4.3.2 and Lemma 4.2.5. The result is important for our later investigation of the algebraic properties of the joint torsion transition numbers.

**Theorem 4.3.3.** *The vertical and the horizontal torsion isomorphism*

$$\begin{aligned} T(v) : \det(H_+(X)) &\rightarrow \det(H_-(X)) & \text{and} \\ T(h) : \det(H_+(X)) &\rightarrow \det(H_-(X)) \end{aligned}$$

*of the odd homotopy exact bitriangle  $(X, v, h)$  agree.*

Let us spell out the implications of Theorem 4.3.3 for the torsion isomorphisms of the columns and rows of the odd homotopy exact bitriangle  $X$ .

To this end, we introduce the  $\mathbb{Z}_2$ -graded homology groups  $H(X_v)$  and  $H(X_h)$  which are defined as the direct sums

$$H(X_v) = H(X_{*1}) \oplus H(X_{*2}) \oplus H(X_{*3}) \quad \text{and} \quad H(X_h) = H(X_{1*}) \oplus H(X_{2*}) \oplus H(X_{3*})$$

of  $\mathbb{Z}_2$ -graded homology groups.

The tensor product of the torsion isomorphisms associated with the columns yields an isomorphism

$$T(X_v) := T(X_{*1}) \otimes T(X_{*2}) \otimes T(X_{*3}) : \det(H_+(X_v)) \rightarrow \det(H_-(X_v))$$

Likewise, the tensor product of the torsion isomorphisms associated with the rows yields an isomorphism

$$T(X_h) := T(X_{1*}) \otimes T(X_{2*}) \otimes T(X_{3*}) : \det(H_+(X_h)) \rightarrow \det(H_-(X_h))$$

Finally, we have a permutation isomorphism

$$\theta = \begin{pmatrix} \theta_+ & 0 \\ 0 & \theta_- \end{pmatrix} : H_+(X_v) \oplus H_-(X_v) \rightarrow H_+(X_h) \oplus H_-(X_h)$$

The next corollary is then a reformulation of Theorem 4.3.3.

**Corollary 4.3.4.** *The torsion isomorphisms of the columns and rows of the odd homotopy exact bitriangle  $X$  agree up to conjugation by the determinant of the permutation matrix  $\theta$ . Thus, we have the identity*

$$T(X_v) = \det(\theta_-)^{-1} \circ T(X_h) \circ \det(\theta_+) : \det(H_+(X_v)) \rightarrow \det(H_-(X_v))$$

*of vertical and horizontal torsion isomorphisms.*

## 5. ALGEBRAIC PROPERTIES

In this section we will prove the triviality result and the multiplicativity result for the joint torsion transition numbers. These results were advertised in the introduction to the paper. Our main tool is the comparison theorem for vertical and horizontal torsion isomorphisms arising from an odd homotopy exact bitriangle. See Section 4.3.

**5.1. Triviality.** Let  $A = (A_1, \dots, A_n)$  be a commuting tuple of linear operators on a vector space  $E$ . Let  $1 \leq i < j \leq n$  be two numbers between 1 and  $n$ . We will assume that the commuting tuple obtained from  $A$  by removing both of the operators  $A_i$  and  $A_j$  is Fredholm. Thus,

$$(ij)(A) = (A_1, \dots, \widehat{A}_i, \dots, \widehat{A}_j, \dots, A_n)$$

is a commuting Fredholm tuple. It then follows from Theorem 3.2.3 that the commuting tuples

$$i(A) = (A_1, \dots, \widehat{A}_i, \dots, A_n) \quad \text{and} \quad j(A) = (A_1, \dots, \widehat{A}_j, \dots, A_n)$$

are Fredholm. We can thus make sense of the joint torsion transition number  $\tau_{i,j}(A) \in \mathbb{F}^*$ . The aim of this section is to prove that  $\tau_{i,j}(A) = 1$ .

We define the odd homotopy exact bitriangle  $X$  by the diagram of odd anti-commuting chain maps

$$(5.1) \quad \begin{array}{ccccccc} K((ij)(A)) & \xrightarrow{A_i} & K((ij)(A))[1] & \xrightarrow{\iota_i} & K(j(A)) & \xrightarrow{\iota_i^* \varepsilon_i^*} & K((ij)(A)) \\ A_j \downarrow & & -A_j \downarrow & & A_j \downarrow & & A_j \downarrow \\ K((ij)(A))[1] & \xrightarrow{A_i} & K((ij)(A)) & \xrightarrow{\iota_i} & K(j(A))[1] & \xrightarrow{-\iota_i^* \varepsilon_i^*} & K((ij)(A))[1] \\ \iota_{j-1} \downarrow & & -\iota_{j-1} \downarrow & & \iota_j \downarrow & & \iota_{j-1} \downarrow \\ K(i(A)) & \xrightarrow{A_i} & K(i(A))[1] & \xrightarrow{\iota_i} & K(A) & \xrightarrow{\iota_i^* \varepsilon_i^*} & K(i(A)) \\ \iota_{j-1}^* \varepsilon_{j-1}^* \downarrow & & -\iota_{j-1}^* \varepsilon_{j-1}^* \downarrow & & \iota_j^* \varepsilon_j^* \downarrow & & \iota_{j-1}^* \varepsilon_{j-1}^* \downarrow \\ K((ij)(A)) & \xrightarrow{A_i} & K((ij)(A))[1] & \xrightarrow{\iota_i} & K(j(A)) & \xrightarrow{\iota_i^* \varepsilon_i^*} & K((ij)(A)) \end{array}$$

and the diagram of odd homotopies

$$\begin{array}{ccccccc} K((ij)(A)) & \xleftarrow{0} & K((ij)(A))[1] & \xleftarrow{\iota_i^*} & K(j(A)) & \xleftarrow{\varepsilon_i \iota_i} & K((ij)(A)) \\ \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 \\ K((ij)(A))[1] & \xleftarrow{0} & K((ij)(A)) & \xleftarrow{\iota_i^*} & K(j(A))[1] & \xleftarrow{-\varepsilon_i \iota_i} & K((ij)(A))[1] \\ \uparrow \iota_{j-1}^* & & \uparrow -\iota_{j-1}^* & & \uparrow \iota_j^* & & \uparrow \iota_{j-1}^* \\ K(i(A)) & \xleftarrow{0} & K(i(A))[1] & \xleftarrow{\iota_i^*} & K(A) & \xleftarrow{\varepsilon_i \iota_i} & K(i(A)) \\ \uparrow \varepsilon_{j-1} \iota_{j-1} & & \uparrow -\varepsilon_{j-1} \iota_{j-1} & & \uparrow \varepsilon_j \iota_j & & \uparrow \varepsilon_{j-1} \iota_{j-1} \\ K((ij)(A)) & \xleftarrow{0} & K((ij)(A))[1] & \xleftarrow{\iota_i^*} & K(j(A)) & \xleftarrow{\varepsilon_i \iota_i} & K((ij)(A)) \end{array}$$

We leave it to the reader to verify the appropriate identities, see Definition 4.1.3. Notice in this respect that the rows and columns are homotopy exact by Theorem 3.3.1. The remaining anti-commutativity relations are in fact satisfied at the level of chain complexes. We will use the odd homotopy exact bitriangle  $X$  to obtain the triviality result. Remark that the homology group  $H(X)$  is finite dimensional by assumption.

**Theorem 5.1.1.** *The joint torsion transition number  $\tau_{i,j}(A) = 1$  is trivial.*

*Proof.* We need to prove that the torsion isomorphisms

$$(-1)^{\mu_i(A)} T_i(A) \text{ and } (-1)^{\mu_j(A)} T_j(A) : \det(H_+(A)) \rightarrow \det(H_-(A))$$

agree. Here the non-negative numbers  $\mu_i(A)$  and  $\mu_j(A) \in \mathbb{N} \cup \{0\}$  are defined in (3.5). We will do this by applying the comparison Corollary 4.3.4. The identifications of determinants which appear are carried out according to the sign convention stated in the beginning of Section 3.

We start by looking at the tensor product of torsion isomorphisms of the rows in  $X$ . We then have the identifications

$$\begin{aligned} T(X_{1*}) \otimes T(X_{2*}) \otimes T(X_{3*}) &= (-1)^{\dim(H_-(j(A)))} T(X_i^{j(A)}) \otimes T(X_i^{j(A)})^{-1} \otimes T(X_i^A) \\ &\cong (-1)^{\dim(H_-(j(A)))} T(X_i^A) \\ &\cong (-1)^{\dim(H_-(j(A)))} T_i(A) \end{aligned}$$

for this tensor product. Remark that the sign on the last odd chain map of the second row is responsible for the sign

$$(-1)^{\dim(H_-(j(A))) + \dim(H_-((ij)(A))) + \dim(H_-((ij)(A)))} = (-1)^{\dim(H_-(j(A)))}$$

Likewise, we look at the tensor product of torsion isomorphisms of the columns in  $X$ . We then have the identifications

$$\begin{aligned} T(X_{*1}) \otimes T(X_{*2}) \otimes T(X_{*3}) &= (-1)^{\text{Ind}((ij)(A)) + \dim(H_-(i(A)))} T(X_{j-1}^{i(A)}) \otimes T(X_{j-1}^{i(A)})^{-1} \otimes T(X_j^A) \\ &\cong (-1)^{\text{Ind}((ij)(A)) + \dim(H_-(i(A)))} T(X_j^A) \\ &\cong (-1)^{\text{Ind}((ij)(A)) + \dim(H_-(i(A)))} T_j(A) \end{aligned}$$

for this tensor product.

Finally, we look at the sign of the permutation

$$\theta : H_+(X_v) \oplus H_-(X_v) \rightarrow H_+(X_h) \oplus H_-(X_h)$$

of homology groups. See Corollary 4.3.4. It can be verified that this sign is given by

$$\text{sgn}(\theta) = (-1)^{\text{Ind}((ij)(A))}$$

From the comparison of vertical and horizontal torsion isomorphisms we then get the identity

$$(-1)^{\dim(H_-(i(A)))} T_j(A) = (-1)^{\dim(H_-(j(A)))} T_i(A)$$

The result of the lemma now follows by the triviality of the Fredholm indices,

$$\text{Ind}(i(A)) = 0 = \text{Ind}(j(A))$$

See Theorem 3.2.3. Indeed, this vanishing result implies that

$$(-1)^{\mu_k(A)} = (-1)^{\dim(H_+(k(A))) \cdot \dim(H_-(k(A)))} = (-1)^{\dim(H_-(k(A)))} \quad k = i, j$$

□

**5.2. Multiplicativity.** Let  $A = (A_1, \dots, A_n)$  and  $B = (B_1, A_2, \dots, A_n)$  be two  $n$ -tuples of commuting linear operators on a vector space  $E$ . The two tuples only differ in the first coordinate. Remark that we do not assume that the linear operators  $A_1$  and  $B_1$  commute. Let us also fix two numbers  $i, j \in \{1, \dots, n\}$ .

We define the product of  $A$  and  $B$  as the  $n$ -tuple of commuting linear operators

$$A \cdot B := (A_1 \cdot B_1, A_2, \dots, A_n)$$

The goal of this section is to prove a multiplicativity relation for the joint torsion transition numbers. To be more precise, we will prove that we have the identity

$$\tau_{i,j}(A \cdot B) = \tau_{i,j}(A) \cdot \tau_{i,j}(B)$$

when the joint torsion transition numbers make sense. Notice that the slightly more general result stated in Theorem 1.0.5 follows by the symmetry property of the joint torsion transition numbers, see Theorem 3.3.5.

An important tool is the relation of Corollary 4.3.4 between vertical and horizontal torsion isomorphisms. In order to apply this result we need to establish the relevant homotopy exact bitriangles. This will be accomplished in the next lemmas.

To begin with, we will establish a link between the Koszul complexes  $K(B)$ ,  $K(A \cdot B)$  and  $K(A)$ . The link is given by the following odd triangle of  $\mathbb{Z}_2$ -graded chain complexes

$$M(A, B) : K(B) \xrightarrow{\nu(A_1)} K(A \cdot B)[1] \xrightarrow{\mu(B_1)} K(A) \xrightarrow{\varepsilon_1^*} K(B)$$

Here the odd chain maps

$$\nu(A_1) : K(A) \rightarrow K(A \cdot B)[1] \quad \text{and} \quad \mu(B_1) : K(A \cdot B)[1] \rightarrow K(B)$$

are given by

$$\nu(A_1) := \varepsilon_1 \varepsilon_1^* + A_1 \varepsilon_1^* \varepsilon_1 \quad \text{and} \quad \mu(B_1) := B_1 \varepsilon_1 \varepsilon_1^* + \varepsilon_1^* \varepsilon_1$$

We would like to show that  $M(A, B)$  is homotopy exact. To this end we define the odd homotopy

$$M^\dagger(A, B) : K(B) \xleftarrow{\varepsilon_1 \varepsilon_1^*} K(A \cdot B)[1] \xleftarrow{\varepsilon_1^* \varepsilon_1} K(A) \xleftarrow{\varepsilon_1} K(B)$$

**Lemma 5.2.1.** *The odd triangle of  $\mathbb{Z}_2$ -graded chain complexes  $M(A, B)$  is homotopy exact.*

*Proof.* We should start by checking that the composition of any two successive odd chain maps in the odd triangle  $M(A, B)$  is chain homotopic to zero with homotopies given by  $M^\dagger(A, B)$ . See Definition 2.3.2. The verifications are all straight forward. However, for the convenience of the reader, we will present the most complicated computation. We have that

$$\mu(B_1) \circ \nu(A_1) = B_1 \varepsilon_1 \varepsilon_1^* + A_1 \varepsilon_1^* \varepsilon_1$$

On the other hand we have that

$$\begin{aligned} d^A \varepsilon_1 + \varepsilon_1 d^B &= A_1 \varepsilon_1^* \varepsilon_1 + B_1 \varepsilon_1 \varepsilon_1^* + \sum_{i=2}^n A_i \varepsilon_i^* \varepsilon_1 + \sum_{i=2}^n A_i \varepsilon_1 \varepsilon_i^* \\ &= A_1 \varepsilon_1^* \varepsilon_1 + B_1 \varepsilon_1 \varepsilon_1^* \end{aligned}$$

This proves that  $\mu(B_1) \circ \nu(A_1) : K(B) \rightarrow K(A)$  is chain homotopic to zero with homotopy given by  $\varepsilon_1 : K(B) \rightarrow K(A)$ .

We should then check the "homotopy decomposition" condition of Definition 2.3.2. Again, everything follows by straight forward computations. In fact, the desired identities are all valid at the level of chain complexes.  $\square$

Remark that a combination of Lemma 5.2.1 and Lemma 2.3.3 yields a proof of the additivity property for Fredholm tuple indices. See Theorem 3.2.4.

Let  $m \in \{2, \dots, n\}$  we will now define a homotopy exact bitriangle  $X(m)$  of  $\mathbb{Z}_2$ -graded chain complexes. It is given by the following anti-commuting diagram of odd chain maps

$$\begin{array}{ccccccc}
 K(m(B)) & \xrightarrow{A_m} & K(m(B))[1] & \xrightarrow{\iota_m} & K(B) & \xrightarrow{\iota_m^* \varepsilon_m^*} & K(m(B)) \\
 \nu(A_1) \downarrow & & -\nu(A_1) \downarrow & & \nu(A_1) \downarrow & & \nu(A_1) \downarrow \\
 K(m(A \cdot B))[1] & \xrightarrow{A_m} & K(m(A \cdot B)) & \xrightarrow{\iota_m} & K(A \cdot B)[1] & \xrightarrow{-\iota_m^* \varepsilon_m^*} & K(m(A \cdot B))[1] \\
 \mu(B_1) \downarrow & & -\mu(B_1) \downarrow & & \mu(B_1) \downarrow & & \mu(B_1) \downarrow \\
 K(m(A)) & \xrightarrow{A_m} & K(m(A))[1] & \xrightarrow{\iota_m} & K(A) & \xrightarrow{\iota_m^* \varepsilon_m^*} & K(m(A)) \\
 \varepsilon_1^* \downarrow & & -\varepsilon_1^* \downarrow & & \varepsilon_1^* \downarrow & & \varepsilon_1^* \downarrow \\
 K(m(B)) & \xrightarrow{A_m} & K(m(B))[1] & \xrightarrow{\iota_m} & K(B) & \xrightarrow{\iota_m^* \varepsilon_m^*} & K(m(B))
 \end{array}$$

and the diagram of odd homotopies

$$\begin{array}{ccccccc}
 K(m(B)) & \xleftarrow{0} & K(m(B))[1] & \xleftarrow{\iota_m^*} & K(B) & \xleftarrow{\varepsilon_m \iota_m} & K(m(B)) \\
 \uparrow \varepsilon_1 \varepsilon_1^* & & \uparrow -\varepsilon_1 \varepsilon_1^* & & \uparrow \varepsilon_1 \varepsilon_1^* & & \uparrow \varepsilon_1 \varepsilon_1^* \\
 K(m(A \cdot B))[1] & \xleftarrow{0} & K(m(A \cdot B)) & \xleftarrow{\iota_m^*} & K(A \cdot B)[1] & \xleftarrow{-\varepsilon_m \iota_m} & K(m(A \cdot B))[1] \\
 \uparrow \varepsilon_1^* \varepsilon_1 & & \uparrow -\varepsilon_1^* \varepsilon_1 & & \uparrow \varepsilon_1^* \varepsilon_1 & & \uparrow \varepsilon_1^* \varepsilon_1 \\
 K(m(A)) & \xleftarrow{0} & K(m(A))[1] & \xleftarrow{\iota_m^*} & K(A) & \xleftarrow{\varepsilon_m \iota_m} & K(m(A)) \\
 \uparrow \varepsilon_1 & & \uparrow -\varepsilon_1 & & \uparrow \varepsilon_1 & & \uparrow \varepsilon_1 \\
 K(m(B)) & \xleftarrow{0} & K(m(B))[1] & \xleftarrow{\iota_m^*} & K(B) & \xleftarrow{\varepsilon_m \iota_m} & K(m(B))
 \end{array}$$

Notice that the vertical triangles are homotopy exact by Lemma 5.2.1 and that the horizontal triangles are homotopy exact by Lemma 3.3.1. This reduces the proof of the homotopy exactness of the bitriangle  $X(m)$  to a verification of the anti-commutativity of several squares. All the calculations involved are however next to trivial and they will therefore not be considered at this place. Remark that the desired identities are actually satisfied at the level of chain complexes.

Suppose that two of the commuting tuples  $m(A), m(B)$  and  $m(A \cdot B)$  are Fredholm. It then follows from Lemma 3.2.4 that the third commuting tuple is Fredholm as well. Furthermore, we see that the homology group  $H(X(m))$  is finite dimensional. We will use the homotopy

exact bitriangle  $X(m)$  to compare the torsion isomorphisms

$$T_m(A) : \det(H_+(A)) \rightarrow \det(H_-(A)) \quad \text{and} \quad T_m(B) : \det(H_+(B)) \rightarrow \det(H_-(B))$$

with the torsion isomorphism

$$T_m(A \cdot B) : \det(H_+(A \cdot B)) \rightarrow \det(H_-(A \cdot B))$$

This is carried out in the next lemma.

**Lemma 5.2.2.** *We have a canonical identification of torsion isomorphisms*

$$\begin{aligned} T(M(A, B)) &\cong (-1)^{\dim(H_-(A \cdot B)) + \mu_m(B) + \mu_m(A \cdot B) + \mu_m(A)} \cdot T_m(B) \otimes T_m(A \cdot B)^{-1} \otimes T_m(A) : \\ &\det(H_+(M(A, B))) \rightarrow \det(H_-(M(A, B))) \end{aligned}$$

Here the exponents  $\mu_m(B)$ ,  $\mu_m(A \cdot B)$  and  $\mu_m(A) \in \mathbb{N} \cup \{0\}$  are defined in (3.5).

*Proof.* We will use the sign convention appearing in the beginning of Section 3.

We start by looking at the tensor product of torsion isomorphisms associated with the rows of the homotopy exact bitriangle  $X(m)$ . We then have the identifications

$$\begin{aligned} T(X(m)_{1*}) \otimes T(X(m)_{2*}) \otimes T(X(m)_{3*}) \\ &= (-1)^{\dim(H_-(A \cdot B))} T(X_m^B) \otimes T(X_m^{A \cdot B})^{-1} \otimes T(X_m^A) \\ &\cong (-1)^{\dim(H_-(A \cdot B))} T_m(B) \otimes T_m(A \cdot B)^{-1} \otimes T_m(A) \end{aligned}$$

of isomorphisms. Remark that the sign on the last odd chain map in the second row is responsible for the sign

$$(-1)^{\dim(H_-(A \cdot B)) + 2 \cdot \dim(H_-(m(A \cdot B)))} = (-1)^{\dim(H_-(A \cdot B))}$$

Next, we look at the tensor product of torsion isomorphisms associated with the columns of the homotopy exact bitriangle  $X(m)$ . We then have the identifications

$$\begin{aligned} T(X(m)_{*1}) \otimes T(X(m)_{*2}) \otimes T(X(m)_{*3}) \\ &= (-1)^{\dim(H_-(X(m)_{*2}))} \cdot T(M(m(A), m(B))) \otimes T(M(m(A), m(B)))^{-1} \otimes T(M(A, B)) \\ &\cong (-1)^{\dim(H_-(X(m)_{*2}))} T(M(A, B)) \end{aligned}$$

of isomorphisms.

We will now examine the sign of the permutation

$$\theta : H_+(X(m)_v) \oplus H_-(X(m)_v) \rightarrow H_+(X(m)_h) \oplus H_-(X(m)_h)$$

of vertical and horizontal homology groups. It can be verified that this sign is given by

$$\text{sgn}(\theta) = (-1)^{\mu_m(B) + \mu_m(A) + \mu_m(A \cdot B) + \dim(H_-(X(m)_{*2}))}$$

The result of the lemma then follows by an application of the comparison Corollary 4.3.4.  $\square$

We will now treat the case where  $m = 1$ .

To this end, we define the odd homotopy exact bitriangle  $X(1)$  by the diagram

$$\begin{array}{ccccccc}
K(1(B)) & \xrightarrow{B_1} & K(1(B))[1] & \xrightarrow{\iota_1} & K(B) & \xrightarrow{\iota_1^* \varepsilon_1^*} & K(1(B)) \\
1 \downarrow & & -A_1 \downarrow & & \nu(A_1) \downarrow & & 1 \downarrow \\
K(1(A \cdot B))[1] & \xrightarrow{A_1 \cdot B_1} & K(1(A \cdot B)) & \xrightarrow{\iota_1} & K(A \cdot B)[1] & \xrightarrow{-\iota_1^* \varepsilon_1^*} & K(1(A \cdot B))[1] \\
0 \downarrow & & -\iota_1 \downarrow & & \mu(B_1) \downarrow & & 0 \downarrow \\
0 & \xrightarrow{0} & K(A)[1] & \xrightarrow{1} & K(A) & \xrightarrow{0} & 0 \\
0 \downarrow & & -\iota_1^* \varepsilon_1^* \downarrow & & \varepsilon_1^* \downarrow & & 0 \downarrow \\
K(1(B)) & \xrightarrow{B_1} & K(1(B))[1] & \xrightarrow{\iota_1} & K(B) & \xrightarrow{\iota_1^* \varepsilon_1^*} & K(1(B))
\end{array}$$

of odd chain maps. This diagram is anti-commutative except for the square

$$\begin{array}{ccc}
K(1(A \cdot B))[1] & \xrightarrow{A_1 \cdot B_1} & K(1(A \cdot B)) \\
0 \downarrow & & -\iota_1 \downarrow \\
0 & \xrightarrow{0} & K(A)[1]
\end{array}$$

The even chain map

$$-\iota_1 \circ (A_1 B_1) : K(1(A \cdot B))[1] \rightarrow K(A)[1]$$

is chain homotopic to zero through the homotopy

$$s := \varepsilon_1 \iota_1 B_1 : K(1(A \cdot B))[1] \rightarrow K(A)[1]$$

The diagram of odd homotopies for  $X(1)$  is given by

$$\begin{array}{ccccccc}
K(1(B)) & \xleftarrow{0} & K(1(B))[1] & \xleftarrow{\iota_1^*} & K(B) & \xleftarrow{\varepsilon_1 \iota_1} & K(1(B)) \\
\uparrow 1 & & \uparrow 0 & & \uparrow \varepsilon_1 \varepsilon_1^* & & \uparrow 1 \\
K(1(A \cdot B))[1] & \xleftarrow{0} & K(1(A \cdot B)) & \xleftarrow{\iota_1^*} & K(A \cdot B)[1] & \xleftarrow{-\varepsilon_1 \iota_1} & K(1(A \cdot B))[1] \\
\uparrow 0 & & \uparrow -\iota_1^* & & \uparrow \varepsilon_1^* \varepsilon_1 & & \uparrow 0 \\
0 & \xleftarrow{0} & K(A)[1] & \xleftarrow{1} & K(A) & \xleftarrow{0} & 0 \\
\uparrow 0 & & \uparrow -\varepsilon_1 \iota_1 & & \uparrow \varepsilon_1 & & \uparrow 0 \\
K(1(B)) & \xleftarrow{0} & K(1(B))[1] & \xleftarrow{\iota_1^*} & K(B) & \xleftarrow{\varepsilon_1 \iota_1} & K(1(B))
\end{array}$$

In order to prove that  $X(1)$  is a homotopy exact bitriangle we need to consider several identities. See Definition 4.1.3. First of all we notice that the homotopy exactness of the rows and columns follows from Lemma 5.2.1 and Theorem 3.3.1. We should then prove that the linear maps

$$vs + sv + ht + th \quad \text{and} \quad hs + sh + vr + rv : H(X(1)) \rightarrow H(X(1))$$

are trivial. It turns out that these maps are already trivial at the level of chain complexes. As an example, let us verify the identity

$$hs + sh + vr + rv = 0 : K(A \cdot B)[1] \rightarrow K(A)[1]$$

On this component we have that

$$\begin{aligned} hs + sh + vr + rv &= -\varepsilon_1 \iota_1 B_1 \iota_1^* \varepsilon_1^* - \iota_1 \iota_1^* + \mu(B_1) \\ &= -B_1 \varepsilon_1 \varepsilon_1^* - \varepsilon_1^* \varepsilon_1 + \mu(B_1) \\ &= 0 \end{aligned}$$

The rest of the vanishing results follows by similar straight forward calculations. Let us suppose that the commuting tuple  $1(A) = 1(B) = 1(A \cdot B)$  is Fredholm. In particular, we get that the homology group  $H(X(1))$  is finite dimensional. We will use the homotopy exact bitriangle  $X(1)$  to compare the torsion isomorphisms

$$T_1(A) : \det(H_+(A)) \rightarrow \det(H_-(A)) \quad \text{and} \quad T_1(B) : \det(H_+(B)) \rightarrow \det(H_-(B))$$

with the torsion isomorphism

$$T_1(A \cdot B) : \det(H_+(A \cdot B)) \rightarrow \det(H_-(A \cdot B))$$

This is carried out in the next lemma.

**Lemma 5.2.3.** *We have a canonical identification of torsion isomorphisms*

$$\begin{aligned} T(M(A, B)) &\cong (-1)^{\dim(H_-(A \cdot B)) + \mu_1(A)} T_1(B) \otimes T_1(A \cdot B)^{-1} \otimes T_1(A) : \\ &\det(H_+(M(A, B))) \rightarrow \det(H_-(M(A, B))) \end{aligned}$$

*Proof.* We will use isomorphisms similar to the sign convention stated in the beginning of Section 3.

Let us look at the tensor product of vertical torsion isomorphisms associated with the homotopy exact bitriangle  $X(1)$ . We then have the identifications

$$\begin{aligned} &T(X(1)_{*1}) \otimes T(X(1)_{*2}) \otimes T(X(1)_{*3}) \\ &= (-1)^{\text{Ind}(1(A)) + \dim(H_-(A))} \cdot T(X(1)_{*1}) \otimes T(X_1^A)^{-1} \otimes T(M(A, B)) \\ &\cong (-1)^{\text{Ind}(1(A)) + \dim(H_-(A)) + \mu_1(A)} \cdot T_1(A)^{-1} \otimes T(M(A, B)) \end{aligned}$$

for this tensor product.

Next, we take a look at the tensor product of horizontal torsion isomorphisms associated with the homotopy exact bitriangle  $X(1)$ . We then have the identifications

$$\begin{aligned} &T(X(1)_{1*}) \otimes T(X(1)_{2*}) \otimes T(X(1)_{3*}) \\ &= (-1)^{\dim(H_-(A \cdot B))} \cdot T(X_1^B) \otimes T(X_1^{A \cdot B})^{-1} \otimes T(X(1)_{3*}) \\ &\cong (-1)^{\dim(H_-(A \cdot B)) + \dim(H_-(A))} T_1(B) \otimes T_1(A \cdot B)^{-1} \end{aligned}$$



for this tensor product. Here we have used that  $\text{Ind}(A) = 0$ . Indeed, the torsion isomorphism  $T(X(1)_{3*})$  is really given by the sign

$$T(X(1)_{3*}) \cong (-1)^{\dim(H_-(A)) \cdot \dim(H_+(A))} = (-1)^{\dim(H_-(A))}$$

Finally, we consider the sign of the permutation

$$\theta : H_+(X(1)_v) \oplus H_-(X(1)_v) \rightarrow H_+(X(1)_h) \oplus H_-(X(1)_h)$$

of homology groups. It can be checked that this sign is given by

$$\text{sgn}(\theta) = (-1)^{\text{Ind}(1(A))}$$

The result of the lemma now follows by an application of the comparison Corollary 4.3.4.  $\square$

We are now ready to prove the main result of this section: The multiplicativity of the joint torsion transition numbers.

**Theorem 5.2.4.** *Suppose that the joint torsion transition numbers in position  $(i, j)$  are well-defined for two of the three commuting tuples  $A$ ,  $B$  and  $A \cdot B$ . Then the joint torsion transition number is well-defined for the third commuting tuple and is linked to the two others by the multiplicativity relation*

$$\tau_{i,j}(A \cdot B) = \tau_{i,j}(A) \cdot \tau_{i,j}(B)$$

*Proof.* The result follows from the definition of the joint torsion transition numbers and by an application of Lemma 5.2.2 and Lemma 5.2.3.  $\square$

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